MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1933 MAY 12

Professor F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Vladimir Guth, Ph.D., Assistant at the National Observatory, Prague, 11 Preslova, Praha xvi, Smichov, Czechoslovakia ;

S. Kawasaki, Mizusawa Observatory, Japan, and 247 Hills Road, Cambridge ;

Leonard Liddicoat, Grinstead, Malvina Avenue, Gravesend, Kent ;

Miss Bohumila Nováková, Ph.D., 4 Velehradská, Praha xii, Czechoslovakia ;

R. W. B. Pearse, A.R.C.S., D.I.C., Ph.D., Assistant Lecturer in Astrophysics, Royal College of Science, S. Kensington, London, S.W. 7 ; and

R. H. Stoy, B.A., Gonville and Caius College, Cambridge.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

Kenneth Davies Hopkins, B.Sc., 105 Turves Green, Northfield, Birmingham (proposed by J. G. Howarth).

Seventy-three presents were announced as having been received since the last meeting, including :—

Bordeaux Observatory, Annales, tome xvii ;

J. Scott, "The Adjustments of an Equatorial Mounting for Astronomical Telescopes."

Dr. V. M. Slipher delivered the George Darwin Lecture, taking as his subject "Spectrographic Studies of the Planets"; after which the President presented him with the Gold Medal of the Society, which had been awarded to him for his spectroscopic researches on planets, stars and nebulæ. [The lecture will be printed in a later number of *Monthly Notices*.]

At the conclusion of the lecture the President said:

It is now my pleasant duty to express on your behalf our thanks to Professor Slipher for the fascinatingly interesting lecture that he has given to us. Fortunately I am able to express our thanks not merely in words but also in concrete form in handing to him the Gold Medal of the Society, duly awarded to him last January by the Council. It has been the custom in the past for the President at the Annual General Meeting to give an address on the researches of the medallist with special reference to the work on which the award of the Gold Medal has been based. The institution of the George Darwin lecture, in which the medallist gives an account of some of his own work at the invitation of the Society, has naturally involved a certain amount of repetition between the addresses of the President and the Medallist, and, as explained to you at the last annual meeting by Dr. Knox-Shaw, the Council has abandoned the Medal address at the annual meeting. That does not mean that the President must present the medal without any remarks in support of the award, but it does mean that he may use the George Darwin lecture to shorten his own address.

The Gold Medal was awarded this year to Professor V. M. Slipher for his spectroscopic researches on planets, stars and nebulæ. Of his work on planets I need say no more now than that it played a very important part in the award. You are in a better position to appreciate that fact after hearing the George Darwin lecture than you would have been, had you merely listened to the President's account of the published work of the medallist on planetary spectra. But I must add that of Professor Slipher's own valuable researches "the half he hath not told you", and I must ask your leave to speak on his work on stellar and nebular spectra.

From among his studies of stellar spectra I single out for mention one outstanding piece of pioneer work, his discussion of the stationary calcium lines in the spectra of a number of stars. When Dr. Hartmann in 1904 first announced the discovery of a fixed line due to ionized calcium in the spectrum of δ Orionis, he made the suggestion that this was due to an intervening cloud of calcium vapour*; he found support for this from the parallel phenomenon of fixed calcium and sodium lines in the spectrum of Nova Persei, and noted that the calcium clouds suggested by both stars were at rest relatively to the 280 stars used by Campbell in his determination of the solar motion.

Our medallist in 1909† made what Dr. J. S. Plaskett, an active worker in this same field, has described as a "most suggestive and penetrating early contribution to the problem of the stationary H and K lines". By studying a number of stars in different regions of the sky—in the constellations Scorpio, Ophiuchus, Orion and Perseus—he was able widely to extend the

* *Ap. J.*, 19, 274, 1904. † *Lowell Obs. Bull.* 51.

region where calcium absorption could be found in stellar spectra showing only a very small velocity relative to Campbell's stars. With some certainty he advanced the hypothesis "that the calcium absorption has its origin in an interposing cloud covering at least certain extensive regions of the sky". It was unfortunate that this paper by Dr. Slipher seems generally to have escaped notice—I find no mention of it, for instance, in a short article by myself on the fixed calcium lines in the *Observatory* in 1913,—but his results in due time received ample confirmation, and his views have now found general acceptance.

Of Dr. Slipher's other contributions to stellar spectroscopy, time will only allow me to mention two. His beautiful spectra of Mira Ceti were used by Professor Alfred Fowler in his detailed comparison with the spectrum of titanium oxide; also in Dr. Slipher we have one of the early workers on the radial velocities of the stars. I mention this because in his case it led on to another pioneer investigation which proved to be of great interest and importance. With a stellar spectrograph adapted for faint objects by the use of a camera lens of short focal length, he developed a most successful technique for the study of the spectra of spiral and diffuse nebulosities.

Following upon the early work of Huggins and Scheiner on the spectrum of the Andromeda nebula, Fath and Max Wolf had established the type of spectrum found in many spiral nebulæ. Dr. Slipher examined the diffuse nebulosities in the Pleiades, in Ophiuchus, in Orion and elsewhere and found spectra related closely to those of the principal stars in their neighbourhood in the sky: he put forward the view that these nebulæ shone by reflected light from the stars. He applied his idea to Hubble's variable nebula (N.G.C. 2261) with its nova-like spectrum shining with light spreading out in successive waves from the stellar apex, and he put forward ideas with a very modern note as to the connection between the bright diffuse nebulæ and the dark absorption nebulæ.

Two other contributions of his to nebular spectroscopy have been of great interest to cosmogonists. The inclinations of spectral lines across the spiral nebulæ were used, as with the planets, to prove and to measure the rotations of the spirals: Dr. Slipher found that the inner portions of these nebulæ were moving out into the arms in the direction of a spring that is being wound up. Again, in the establishment of the high velocities of recession of the spiral nebulæ, another piece of pioneer work of great importance was carried out by Dr. Slipher. In a series of studies of the radial velocities of these island galaxies he laid the foundations of the great structure of the expanding Universe, to which others, both observers and theorists, have since contributed their share. If cosmogonists to-day have to deal with a Universe that is expanding in fact as well as in fancy, at a rate which offers them special difficulties, a great part of the initial blame must be borne by our medallist.

Dr. Slipher, in handing to you this medal as a token of recognition by the Royal Astronomical Society of the fine quality and pioneering nature of your investigations, I gladly couple with it the best wishes of the Fellows for much happiness and success in your future work and for the long-continued prosperity of the Observatory which you so ably direct.

THE SPECTROSCOPIC DETERMINATION OF STELLAR ROTATION AND ITS EFFECTS ON LINE PROFILES.

J. A. Carroll, M.A., Ph.D., Professor of Natural Philosophy in the University of Aberdeen.

1. In a previous communication to the Society * I discussed the effects of axial rotation of a star on the observed form of its absorption lines, and formulated the integral equation connecting the observed form of the line with the "true" form. By the true form is meant the form which would be observed by an observer at rest with respect to axes fixed in the star, or, in other words, the form of the line yielded by an element of the disc moving only across the observer's line of sight.

The problem existing is twofold. Firstly, to determine the rotational speed of the star, or, more correctly, its equatorial velocity in the line of sight, and secondly, given this quantity, to deduce from the observed line the form of the line emitted by an element of the stellar surface.

At the time of my earlier discussion few measurements of the shapes of stellar lines existed. As it was not thought that stellar rotational speeds exceeded some 20–30 km./sec. at the star's equator, I obtained a solution of the equations valid only when the distortion of the line due to rotation is slight, and dealt for the most part with the change in the central intensity of a stellar absorption line. Since then many observations have been made of the profiles † of stellar absorption lines, and in particular the observations of Struve and Elvey have shown ‡ that velocities of the order of ten times the limit formerly supposed to exist are quite likely.

Shajn and Struve ‡ showed that in rapidly rotating stars the lines are often of a broad shallow form to which the name "dish-shaped" was very appropriately attached. Elvey showed, by working out numerically from a line of ordinary shape the observed form due to various rotational speeds, that this is the shape to be expected when the speed is high. Strictly speaking, as will be shown below, this shape is characteristic not specifically of high rotational speeds but of great distortion of the original line, though since stellar lines are not usually very narrow a high speed will as a rule be required to produce the effect.

Struve and Elvey have made estimates of rotational speeds for a number of stars, by a process of matching the observed form of the line with shapes calculated from an assumed original or true shape at various speeds. It

* *M.N.*, 88, 548, 1928.

† At the 1932 Meeting of the I.A.U. it was agreed informally, after discussion by the Solar Physics Commission, that the use of the word "contour" to denote the shape of a spectrum line is undesirable, and the word "profile" was recommended as more suitable.

‡ Shajn and Struve, *M.N.*, 89, 222, 1929; Elvey, *Ap. J.*, 70, 152, 1929; 71, 221, 1929; Struve, *ibid.*, 72, 1, 1930.

will appear later that this method gives, under certain restrictions, a fair estimate of rotational speed, assuming this to be the cause of the deformation ; the main defect of the method is that it assumes a knowledge of the original line we do not possess.

In view of these discoveries I re-examined the problem and have obtained a complete solution of the equations, leading to several interesting results, and some valuable formulæ for use in various cases occurring in practice.

Since two unknown quantities are involved in the production of the observed line, namely, the original form of the line and the rotational speed of the star, it is to be expected that they cannot be determined individually without either some further experimental evidence or some assumption in addition to a knowledge of the observed line. Surprisingly, this expectation turns out, as will be shown, to be wrong ; and theoretically, if only an observed line *is* distorted by stellar rotation, it is possible to discover this fact and to find both the rotational speed of the star and the original form of the line.

In my earlier discussion I pointed out that *a priori* knowledge of relative central intensities of lines would suffice, but, under conditions existing then and now, this is in practice of no service. To make any progress some additional assumption appeared to be necessary. Elvey and Struve assume a knowledge of the original form of the line. This is of course adequate, but is obviously to be avoided if possible. In the present discussion it is shown that far less sweeping assumptions suffice to determine the rotational speed (with considerable precision, if it be large) and hence the original form of the line. This has two advantages : it eases the practical calculation of the quantities required, and, what is more important, makes the belief that the effects observed are due to rotation of the star much more probable.

In the discussion of the problem which follows I shall show how it is possible to find out whether an observed line has been distorted by rotation of the emitting star and how the rotational speed may be found, then certain other devices for finding the rotational speed will be described. These latter involve assumptions not strictly necessary theoretically, but useful in certain cases occurring in practice, and facilitating calculation and examination of particular observed lines. The problem of computing the "original" form of the line will be treated and practical methods described. It will be convenient, however, since some of the results there obtained will be required later, to discuss the solution of the integral equation first, *i.e.* how to find the original line given the observed line and the rotational speed. This will be followed by the analysis of line shape to test for rotational distortion, and the next part of the paper will consist of numerical examples in illustration and test of the theory developed.

2. *The Integral Equation for a Spectrum Line in a Rotating Star.*—It is unnecessary to repeat the algebra of the first paper * leading to the equation

$$O(\zeta) = \frac{2}{\pi}\int_{-1}^{+1} I(\zeta + \beta t)\sqrt{1 - t^2}\,dt, \qquad (2.1)$$

* *M.N.*, 88, 553, 1928.

where $I(\zeta)$ is the "true" shape of a line and $O(\zeta)$ the observed shape, and β is $\frac{V \sin \theta}{c}$, where V is the equatorial velocity of the star, θ the inclination of its axis of rotation to the normal to the line of sight and c the velocity of light. ζ is $\frac{\lambda - \lambda_m}{\lambda_m}$, where λ_m is the wave-length of the centre of the line and λ the wave-length at any other point. It should be noted that the introduction of this quantity has the effect of removing the irrelevant λ_m from the equations and dealing only with the actual shape of the line, ignoring its position, which only enters as a sort of scale factor. This simplification is important in the treatment, as without it the solution would be much more complicated algebraically. This variable, which might be termed profile distance or intrinsic wave-length, is also the one naturally occurring in theoretical formulations of line profiles, in Unsold's formula for example.

The next step is the generalisation of this equation, obtained originally for a spherical star with no darkening towards the limb, rotating as a rigid body, at least on the surface.

It is hardly practicable to allow for variation of rotational speeds at different latitudes in the present state of knowledge. It seems unlikely that the effect would be great, but we may in any event regard β as an average value. If a law giving the variation of rotation with latitude be known it can readily be inserted without much complication, but, unless it amounts to a difference of angular velocity of over 10 per cent. from equator to pole, will produce inappreciable disturbing effects so far as we are concerned. It is easily seen that the equation for a star which is an ellipsoid of revolution rotating about its axis of figure is identical with (2.1), since the velocity in the line of sight is purely a function of the x co-ordinate on the disc where axes Ox, Oy are taken through the centre of the disc, perpendicular to and in, respectively, the plane through the axis of rotation and the line of sight.

The equation is in fact

$$O(\zeta) = \frac{2}{\pi ab} \int_{-a}^{+a} I\left(\zeta + \frac{\omega \sin \theta x}{c}\right) b \sqrt{a^2 - x^2}\, dx,$$

where the elliptical central section of the star is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and on putting $x = at$ this becomes

$$O(\zeta) = \frac{2}{\pi} \int_{-1}^{+1} I(\zeta + \beta t) \sqrt{1 - t^2}\, dt,$$

where β has the same meaning as before, and V is the equatorial velocity of the ellipsoidal star, of semi-major axis a.

The effect of darkening is definite but slight, merely altering the coefficients in the solution somewhat. In the first paper a law of darkening

$\frac{1}{5}(2+3\cos\theta)$ is assumed, and since this is known to be a good approximation it will be retained in this work and the appropriate coefficients calculated. It will be further assumed that the law of darkening on an elliptical disc leads to the same equation, namely,

$$O(\zeta)=\frac{5}{4\pi}\int_{-1}^{+1} I(\zeta+\beta t)\left\{\frac{8\sqrt{1-t^2}+3\pi(1-t^2)}{10}\right\}dt. \qquad (2.2)$$

It should be noted that it is of no importance whether the profile be that of an emission line or an absorption line; in either case $I(\zeta)$ is simply the intensity relative to $I(0)$ on some suitable scale; if the line be an absorption line $I(0)$ can be taken as the height of $I(\zeta)$ relative to the continuous background, *i.e.* 1 – the residual central intensity.

Note on the "Law of Darkening."—The law used in (2.2) is the approximate solar law for white light, *i.e.* integrated radiation. It is used here as a "combined law of darkening" merely for purposes of comparison with no darkening, and does not imply constancy of line profile over the disc or that monochromatic radiation is not being considered. The equation (2.2) is arrived at in the manner indicated in my first paper, p. 549, as follows: Allowing for change of background intensity, line intensity and line profile we need to consider an expression $s(\theta)I(\zeta,\theta)$, where $s(\theta)$ represents the darkening of the continuous spectrum, and $I(\zeta,\theta)$ the change in the relative intensity of the line and its change of form over the disc, θ being 0 at the centre and $\pi/2$ at the limb. The only accurate information about the behaviour of $I(\zeta,\theta)$ is that obtained by Professor H. H. Plaskett (*M.N.*, **91**, 870, 1931), and his formula is complicated. It seemed to me idle to complicate this discussion by an attempt to include an accurate expression for $I(\zeta,\theta)$, at least until more information about its importance in this connection is available from the reduction of observations. Some allowance for this factor can be made, however, by supposing $I(\zeta,\theta)$ can be split up into $a(\theta)\,.\,I(\zeta)$. This implies a change in the "depth" of the line from centre to limb, its form remaining unchanged. This is certainly better than taking $a(\theta)$ constant, and an expression of the form $a(\theta)=\dfrac{a_1+b_1\cos\theta}{a_1+b_1}$ allows of a fair approximation to the truth. Thus $I(\zeta,\theta)$ becomes

$$\frac{a_1+b_1\cos\theta}{a_1+b_1}\,.\,I(\zeta)$$

as a first approximation, or we may assume that $s(\theta)a(\theta)$ has an expansion of the form $\dfrac{a+b\cos\theta+c\cos 2\theta+\ldots}{a+b+c+\ldots}$, and this is the procedure followed here, where the first two terms only are retained and the values $a=2$, $b=3$ adopted provisionally.

If it turns out that the observational material permits the law of darkening to be tested we can then try more elaborate expressions, and since the test can be performed on individual lines the "continuous" and "monochromatic" parts may even be separable. As will be seen, at present the

indications are that the test of the law is possible, but it is premature to attempt to fix better values for a and b.

As will be clear from § 3 and § 6.1, it is to some extent a matter of indifference which, in the integral equation (3.1), is regarded as the unknown function; we can assume a law of darkening and find β and $I(\zeta)$, or we can assume $I(\zeta)$ and find β and the law of darkening or change of form. At present the former is the proper procedure, later we may be in a position to assume $I(\zeta)$ with sufficient certainty to calculate the other factor.

3. *The Solution of the Integral Equation.*—The general form of equation (2.1) is

$$O(z) = a\int_{-1}^{+1} I(z+\beta t)g(t)dt, \tag{3.1}$$

where g is any given function of t, and

$$\frac{1}{a} = \int_{-1}^{+1} g(t)dt. \tag{3.2}$$

Let $K(u)$ be defined by the relation

$$K(u) = uI(u). \tag{3.3}$$

Then on multiplying both sides of (3.1) by e^{-zx} we obtain

$$e^{-zx}O(z) = a\int_{-1}^{+1} \frac{e^{-zx}}{z} \cdot z\frac{K(z+\beta t)}{z+\beta t}g(t)dt. \tag{3.4}$$

Let C be a contour in the complex (z) plane such that all singularities of O and K lie inside it, or a line $c-\iota\infty$ to $c+\iota\infty$ such that they all lie to the left of it, then

$$\frac{1}{2\pi\iota}\int_c e^{-zx}O(z)dz = \frac{a}{2\pi\iota}\int_c\int_{-1}^{+1}\frac{e^{-zx}}{z} \cdot z\frac{K(z+\beta t)}{z+\beta t}g(t)dt. \tag{3.5}$$

Let

$$\left.\begin{aligned}\frac{1}{2\pi\iota}\int_c e^{-zx}O(z)dz &= \phi(x),\\ \frac{1}{2\pi\iota}\int_c e^{-zx}\frac{K(z)}{z}dz &= \psi(x).\end{aligned}\right\} \tag{3.6}$$

Then

$$\frac{1}{2\pi\iota}\int_c \frac{e^{-zx}}{z} \cdot z\frac{K(z+\beta t)}{z+\beta t}dz = e^{-\beta tx}\psi(x). \tag{3.7}$$

Thus

$$\phi(x) = a\psi(x)\int_{-1}^{+1} e^{-\beta tx}g(t)dt = \psi(x)G(\beta x), \tag{3.8}$$

where

$$G(u) = a\int_{-1}^{+1} e^{-ut}g(t)dt. \tag{3.9}$$

Thus

$$\psi(x) = \frac{1}{G(x)}\phi(x). \tag{3.91}$$

Now in virtue of (3.6) by the Bromwich-Wagner theorem

$$\left.\begin{aligned} O(u) &= \int_0^\infty e^{-ux}\phi(x)dx \\ \text{and} \qquad I(u) &= \frac{K(u)}{u} = \int_0^\infty e^{-ux}\psi(x)dx, \end{aligned}\right\} \tag{3.92}$$

so that finally on multiplying (3.91) by e^{-zx} and integrating from 0 to ∞

$$I(u) = \int_0^\infty e^{-ux}\frac{1}{G(\beta x)}\phi(x)dx \tag{3.93}$$

$$= \frac{1}{2\pi\iota}\int_0^\infty e^{-ux}\frac{1}{G(\beta x)}\int_c e^{-zx}O(z)dzdx. \tag{3.94}$$

This is the solution of equation (3.1), as may be verified by direct substitution. In effect this procedure amounts to a symbolic solution where z in equation (3.1) is regarded as the symbolic operator.* We have retained the full detail of the complex variable theory here, since (*a*) it appears inevitably in the result (3.94), and (*b*) it is of service in discussing any restrictions necessary on the functions employed, *e.g.* to secure convergence or in transforming (3.94) by deforming the contour C.

The solution as it stands in equation (3.94) is useless, since the observed quantity $O(z)$ is to be integrated over a contour in the complex plane, while it is known only numerically for real values of the argument.

We may note here a result that will be of use later and is of general value as a test of approximate formulæ, namely, the areas of the original and the distorted profiles are the same, *i.e.*

$$\int_{-\infty}^\infty O(\zeta)d\zeta = \int_{-\infty}^\infty I(\zeta)d\zeta = A\,; \tag{3.95}$$

this can be seen directly by integration of equation (2.2), or is otherwise obvious considering the fact that the total absorption cannot be altered by rotation.

Two modes of procedure are open to us at this stage. We may be able to reduce (3.94) to a form involving knowledge of $O(z)$ for real values of z only. This will depend on the form of $G(x)$; we shall see presently that this reduction can be effected in two ways that are of service in practice. Otherwise we may seek a function $O(z)$, which represents the observed quantity over the real axis, and is of a form amenable to treatment by equation (3.94), so that the complex integration can be carried out and in effect $\phi(x)$ found in terms of the observed quantities, thus reducing the determination of $I(u)$ to the evaluation of (3.93), which can always be done since $G(x)$ is known.

There exists a method due to Schwarzschild † for representing over the

* In Carson's method, as developed by Van der Pol and Niessen, $zO(z)$ is the image of $\phi(x)$, and $K(z)$ the image of $\psi(x)$.

† *Astr. Nach.*, 4422 and 4557.

whole complex plane a function known only for real values of the argument, but it is troublesome to apply and involves rather precise knowledge of the function to be represented. In view of the special form that $O(\zeta)$ has in the problem of stellar-line profiles, the following is suggested as a convenient method. $O(\zeta)$ is zero or negligible everywhere except over a small range of the argument near $\zeta = 0$, and in this range it is finite, single valued and with at most one or two maxima and minima. Crudely speaking, it is some sort of flattened-out error curve. It can therefore be represented to any accuracy desired by an expression of the form

$$O(\zeta) = \frac{O_0}{1 + a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n}, \tag{3.96}$$

where a_1, a_2 . . . may be calculated directly from the observations (by solving a set of n linear simultaneous equations of the form

$$\frac{O_0}{O(\zeta)} = 1 + a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n$$

for n selected values of ζ). $O(z)$ can have no poles on the real axis and must have a set of poles at conjugate complex points which are found from the equation $1 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$. Thus $O(z)$ can be broken up into partial fractions of the form

$$O(z) = \Sigma \frac{f(z)}{(z-\gamma)^2 + \delta^2}, \tag{3.97}$$

allowing immediate complex integration and yielding terms of the form $\begin{matrix}\sinh \\ \cosh\end{matrix} mx \begin{matrix}\cos \\ \sin\end{matrix} nx$ to be inserted in (3.93). This method (which we will call Method I) is applied in § 6.2. Its accuracy depends on the number of terms used in equation (3.96), which in turn depends on the accuracy of our knowledge of $O(\zeta)$. In any event, paucity of knowledge of the form $O(\zeta)$ will impair the determination of $I(\zeta)$, but this method is not very exacting as will be seen from the example. Against it must be set the considerable labour involved if great accuracy is wanted, and also the fact that for small values of β the integrals of the oscillating functions converge so slowly as to be impractical for values of u near the centre of $I(u)$.

For the convenience of anyone wishing to try this method I give the formula for the polynomial of highest even degree usable in practice, namely, the sixth.

If

$$O = \frac{O_0}{1 + Ap^2 + Bp^4 + Cp^6},$$

A, B, C are found from the observations for three selected values of p, and the expression put in the form

$$O = O_0 \cdot \frac{N}{(p^2 + a^2)(p^4 - b^2p^2 + c^2)},$$

where a^2 is found as the only positive root of $1 - Ap^2 + Bp^4 - Cp^6 = O$. This is factorised as $(p^2+a^2)(\overline{p+\alpha}^2+\beta^2)(\overline{p-\alpha}^2+\beta^2)$, where $\alpha^2+\beta^2=c$, $\alpha^2-\beta^2=b^2/2$, and after splitting into partial fractions and integrating we have

$$\frac{1}{A}\left\{\frac{\sin ax}{a}+\frac{\beta^2-a^2-3\alpha^2}{2\alpha(\alpha^2+\beta^2)}\sinh \alpha x\cos \beta x+\frac{\alpha^2+a^2-3\beta^2}{2\beta(\alpha^2+\beta^2)}\cosh \alpha x\sin \beta x\right\},$$

which, to give $I(u)$, is to be multiplied by $e^{-ux}\dfrac{\beta x}{2I_1(\beta x)}$ (here β means $\dfrac{V\sin\theta}{c}$, the rotational speed) and integrated between the limits 0 and ∞.

Method II.—Another method is to try to find some amenable functional form for $O(\zeta)$ which, with but few disposable constants, will fit a good many observed curves.

Such an empirical formula is

$$O = O_0 e^{b-\sqrt{b^2+a^2z^2}}, \tag{3.98}$$

containing three constants only, which has been found to fit several observed curves well. We require then

$$\frac{1}{2\pi\iota}\int_c e^{zx}\cdot O_0 e^{b-\sqrt{b^2+a^2z^2}}dz,$$

which has branch points of the integrand at $t=\pm\iota$,

where $$t=\frac{az}{b} \quad \text{and} \quad \frac{O_0e^b}{2\pi\iota}\frac{b}{a}\int_c e^{\frac{bx}{a}t-b\sqrt{1+t^2}}dt$$

then reduces to

$$\frac{O_0be^b}{2\pi a}\sum_{r=0}^{\infty}\mathrm{J}_{\frac{r+1}{2}}\left(\frac{bx}{a}\right)\frac{\Gamma\left(\frac{r}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{bx}{a}\right)^{\frac{r+1}{n}}} \tag{3.981}$$

which can be inserted in (3.93).

The integrals should be fairly rapidly convergent.*

The most satisfactory methods are these depending on a reduction of (3.94) to expressions involving $O(\zeta)$ as known numerically only. These fall into the following classes: (*a*) where $G(u)$ is of such a form that $\int_0^\infty \frac{e^{-au}}{G(u)}du$ can be evaluated in functional form and where the integral possesses singularities that allow the complex integral to be reduced (in this class fall the use of approximations to $G(u)$ which admit of simple reduction), (*b*) expansions of $\frac{1}{G(u)}$ admitting of term-by-term integration and consequent reduction of the complex integral yielding a solution in series.

* It is probably simpler not to use the series of Bessel functions, but to evaluate the real form of the integral numerically as it stands with the given values of a and b.

In our particular example of the equation (3.94) the second alternative is the most useful one and may be combined, as will be seen, with the first. We shall deal with two cases.

Method III, where $\frac{1}{G(u)}$ *can be expanded in a Maclaurin series of powers of* u. Let

$$\frac{1}{G(u)} = a_0 + \Sigma a_n u^n. \tag{3.991}$$

Then (3.94) reduces to terms of the form

$$\frac{a_n}{2\pi i}\beta^n \int_0^\infty e^{-ux} x^n \int_c e^{-zx} O(z)\,dz, \tag{3.992}$$

which is immediately integrable as

$$a_n \frac{\beta^n}{n!} O_n(z),$$

where $O_n(z)$ is the nth differential coefficient of $O(z)$. Thus the solution is found in the form

$$I(\zeta) = a_0 O(\zeta) + \Sigma a_n \frac{\beta^n}{n!} O_n(\zeta). \tag{3.993}$$

It is equivalent to the procedure followed in my first paper, and is of rather restricted application, in effect applying only to cases where the distortion of the original line is slight.

If $\frac{1}{G(u)}$ cannot be expanded in the required form over the whole range $0 - \infty$, the method fails unless $\int_0^\infty \frac{e^{-ux}\phi(x)}{G(\beta x)}dx$ can be replaced by $\int_0^r \frac{e^{-ux}\phi(x)}{G(\beta x)}dx$ where $r < R$, the radius of convergence of the expansion of $\frac{1}{G(u)}$.

It is worth remarking here that the coefficients in this solution, and indeed the solution itself when valid, may be most readily obtained thus from the original equation (3.1). If it be legitimate to expand $I(\zeta + \beta t)$ in a Taylor series, *i.e.* in effect, if β be small compared to ζ_0 where $I(\zeta) = 0$ if $|\zeta| \geqslant \zeta_0$, then

$$O(\zeta) = a\sum_{n=0}^{\infty} \int_{-1}^{+1} I_n(\zeta) \frac{\beta^n}{n!} t^n g(t)\,dt$$

$$= \Sigma a_n \frac{\beta^n}{n!} I_n(\zeta),$$

where

$$a_n = a\int_{-1}^{+1} t^n g(t)\,dt,$$

whence

$$O_m(\zeta) = \Sigma \frac{a_m \beta^n}{n!} I_{m+n}(\zeta).$$

And the solution of this infinite set of linear equations for $I(\zeta)$ in terms of $O(\zeta)$ and its differential coefficients is *

$$I(\zeta) = \Sigma \mu_n \beta^n O_n(\zeta),$$

where the μ's and the α's are connected by the recurrence formula;

$$\mu_{2n} + \mu_{2n-1} m_2 + \ldots + \mu_2 m_{2n-2} + m_{2n} = 0,$$

where

$$m_r = \frac{\alpha_r}{r!}.$$

Method IV.—If $\frac{1}{G(u)}$ has a suitable asymptotic expansion the equation may integrate out term by term and yield an asymptotic series for $I(\zeta)$ in terms of $O(\zeta)$ and its derivatives.

This is the most powerful mode of attack on the problem, and in the special case we are considering yields the most useful and comprehensive formulæ.

4. *An Asymptotic Expansion for* I(ζ).—Starting from equation (2.2) the function $G(u)$ of § 3 is by (3.9)

$$G(u) = \frac{5}{4\pi}\int_{-1}^{+1} e^{ut}\left\{\frac{8\sqrt{1-t^2}+3\pi(1-t^2)}{10}\right\}dt$$

$$= \frac{1}{2u^3}\left\{2u^2 I_1(u) + 3u\cosh u - 3\sinh u\right\}. \qquad (4.1)$$

This has the asymptotic expansion

$$G(u) \sim \frac{e^u}{u\sqrt{2\pi u}}\left\{1 + \Sigma(-)^r \frac{(4-1^2)(4-3^2)\ldots(4-\overline{2r-1}^2)}{r!\,2^{3r}\,.\,u^r}\right\} + \frac{3}{4}\frac{e^u}{u^2}\left\{1 - \frac{1}{u}\right\}. \qquad (4.2)$$

Substituting in (3.94), we obtain (on integrating first with respect to x) a series of integrals of the form $\int_0^\infty x^{\frac{n}{2}} e^{-ax} dx$ immediately reducible to Gamma functions, and leading to a set of complex integrals of the form

$$\frac{1}{2\iota}\int_c \frac{O(z)}{(u+\beta-z)^{\frac{n}{2}}} dz,$$

where $R(z) < u + \beta$. These are evaluated by deforming the contour C into one of Hankel's type, which we are at liberty to do if $O(z)$ is zero for all sufficiently large values of z, as is the case.

When n is even Cauchy's theorem applies directly; if n is odd we make use of the following theorem.†

* Carroll, *M.N.*, **90**, 588, 1930, where a set of equations of this form is solved.

† Cauchy, *Exerc. de Math.*, **2**, 91, 1827; Saalschüty, *Zs. für Math. u. Phys.*, **32**, 1887; **33**, 1888; Dixon, *Mess. of Math.*, **33**, 176, 1904. I am indebted to Dr. Copson of St. Andrews for these references.

If $f(x)$ is finite and single valued at all points of the positive real axis then

$$\frac{1}{2\iota \sin a\pi}\int_H (-z)^{a-1}f(z)dz = \int_0^\infty t^{a-1}\left\{f(t)-f(0)-tf'(0)\ \ldots\ -\frac{t^K}{K!}f_K(0)\right\}dt$$

$$= \frac{1}{a(a+1)\ \ldots\ (a+K)}\int_0^\infty t^{a+K}f_{K+1}(t)dt, \qquad (4{\cdot}3)$$

where $f_r(t)$ is the r^{th} differential coefficient of f with respect to t, and K is a positive integer defined by

$$K < |\ a\ | < K+1$$

and H is a contour starting and ending at $+\infty$ encircling the origin once clockwise.

Applying this to our integrals we have, if $a = u - z$,

$$\int_0^\infty \frac{e^{-ax}}{G(\beta x)}dx = \frac{1}{\beta}\int_0^\infty \frac{e^{-\frac{a}{\beta}x}}{G(x)}dx$$

$$\sim \frac{\sqrt{2\pi}}{\beta}\int_0^\infty \frac{x^{\frac{a}{\beta}}e^{-\left(\frac{a}{\beta}+1\right)x}}{\left\{1+\frac{3\sqrt{2\pi}}{4}\cdot\frac{1}{x^{\frac{1}{2}}}-\frac{3}{8}\frac{1}{x}-\frac{3\sqrt{2\pi}}{4}\frac{1}{x^{\frac{3}{2}}}-\frac{15}{128}\frac{1}{x^2}\ \ldots\right\}}dx$$

$$= \frac{\sqrt{2\pi}}{\beta}\int_0^\infty e^{-\left(\frac{a}{\beta}+1\right)x}x^{\frac{a}{\beta}}\left\{1-\frac{3\sqrt{2\pi}}{4}\frac{1}{x^{\frac{1}{2}}}+\left(\frac{9\pi+3}{8}\right)\frac{1}{x}+\frac{6-27\pi}{32}\sqrt{2\pi}\cdot\frac{1}{x^{\frac{3}{2}}}\right.$$

$$\left.+\frac{162\pi^2-126\pi+33}{128}\frac{1}{x^2}\ \ldots\right\}dx.$$

Yielding—

1st Term:

$$-8\left(\frac{\beta}{2}\right)^{\frac{3}{2}}\int_0^\infty p^{\frac{1}{2}}O_3(p+u+\beta)dp.$$

2nd Term:

$$+\frac{3\pi\beta}{2}O_1(u+\beta).$$

3rd Term:

$$+\frac{9\pi+3}{2}\left(\frac{\beta}{2}\right)^{\frac{1}{2}}\int_0^\infty p^{\frac{1}{2}}O_2(p+u+\beta)dp.$$

4th Term:

$$+\frac{27\pi-6}{16}\cdot\pi O(u+\beta).$$

5th Term:

$$-\frac{162\pi^2-126\pi+33}{64}\left(\frac{2}{\beta}\right)^{\frac{1}{2}}\int_0^\infty p^{\frac{1}{2}}O_1(p+u+\beta)dp,$$

$+$, etc. (4·4)

It should be noted that only values of $O(\zeta)$ and its differential coefficients

are involved at points β or more away from the origin, thus on reference to fig. 2 it is seen that only one-half of O or I is involved. If u be $+ve$ the effect of $I(u)$ for $-ve$ values is zero for all points of $O(p)$ where $p \geqslant \beta$, so that it is unnecessary to assume any symmetry in O or I; the two parts are evaluated separately if they differ.

Method V.—Another possibility, depending on the results of § 5.1, is to compute $\int_{-\infty}^{\infty} O(t) \cos u(t-\zeta)dt$ and divide by the function in brackets in (5.13), thus obtaining $\int_{-\infty}^{\infty} I(t) \cos u(t-\zeta)dt$ and hence (on completing the Fourier Integral) $I(\zeta)$. This method is excellent in numerical tests (see § 6) and seems serviceable in practice (see *M.N.*, **93**, 516), but is likely to be troublesome if the observed curve is not accurately known, owing to the singularities introduced when the zeros of $\int_{-\infty}^{\infty} O(t) \cos utdt$ do not coincide with the zeros of $G(u\beta)$.

I expect this, however, to be the best and simplest practical solution of the problem.

5. *The Determination whether an Observed Line is Distorted due to Rotation of the Star, and the Evaluation of the Rotational Speed.*—In the first section of this paper it was stated that it could be discovered whether a given observed line is distorted by rotational effects, and also, if it is, that the rotational speed can be determined, all without knowledge, in addition to that of the observed profile.

The reason for this surprising possibility is that the observed line cannot (if it is distorted by rotation) have any arbitrary shape, but is in part determined by its mode of production; and I shall show how this factor may be disentangled from the factor due to the original (and, so far as we are concerned, arbitrary) form of the line.

To be strictly accurate, we cannot in fact prove that a given observed line profile is due to rotation, though we may be able to prove that it is not; it is of course possible that, by accident, a given observed line not due to rotation has in it just those elements of shape which rotation could have provided, but it will be appreciated from what follows that this is not a very likely contingency, and when we find agreement between two different determinations of rotational speed for one line, and *a fortiori* when two or more lines show the characteristics to be expected of lines produced by rotation with concordant values of the rotational speed, then we may feel convinced that the star is indeed rotating.

Put in its crudest possible form the test resembles that of Elvey and Struve, who recognised the feature to which they gave the name "dish shape" in lines broadened by rotation, but this rough eye-estimate of shape conveys but a scant impression of the analytical test to be described, and affords no direct value for β without assuming the original line to be known completely, as does this method.

5.1. *The Test for Rotational Distortion.*—We can represent both the

original line $I(\zeta)$ and the observed line $O(\zeta)$ by means of Fourier Integrals, in the usual manner, thus

$$\pi f(x) = \int_0^\infty \int_{-\infty}^\infty \cos u(t-x) f(t) dt du. \tag{5.11}$$

The part $\int_{-\infty}^{\infty} \cos u(t-x) f(t) dt$ of the double integral represents the contribution to the function of the terms with a "frequency" $u/2\pi$, *i.e.* considered as a function of u it gives the amplitude spectrum of the function.

Now if $I(\zeta)$ and $O(\zeta)$ be represented by Fourier Integrals in equation (2.2), the resulting spectrum of $O(\zeta)$ must be the same, whether derived directly from the observations or from $I(\zeta)$ by the right-hand side of (2.2). But the right-hand side of (2.2) gives a spectrum with zeros not merely at certain points fixed by $I(\zeta)$, but in addition at other points fixed not by $I(\zeta)$ but by β and the form of $g(t)$, *i.e.* the law of darkening and the fact that rotation is at work leading to the particular form of equation (2.2). There may be other zeros depending on $I(\zeta)$ as I have said, but the point is we can calculate where the "rotational" zeros must come, and since the spectrum can be found directly from the observations $O(\zeta)$, we have only to find it and search among its zeros for ones to satisfy the given conditions.

The procedure is best described by simple illustration.

In the Fourier Integral for $O(\zeta)$ we have the expression

$\int_{-\infty}^{\infty} O(t) \cos u(t-\zeta) dt$, and we are thus led to consider

$$\int_{-\infty}^{\infty} O(t) \cos utdt. \tag{5.12}$$

The other term, $\int_{-\infty}^{\infty} O(t) \sin utdt$, vanishes if $O(t)$ be symmetrical, but it is unnecessary to consider this integral, and we do not assume $O(t)$ an even function, it is the evenness of $g(t)$ that matters.

Now in virtue of (2.2) we have

$$\begin{aligned}
\int_{-\infty}^{\infty} O(t) \cos utdt &= \frac{5}{4\pi} \int_{-\infty}^{\infty} \int_{-1}^{+1} I(t+\beta x) \left\{ \frac{8\sqrt{1-x^2} + 3\pi(1-x^2)}{10} \right\} dx \cos utdt \\
&= \frac{5}{4\pi} \int_{-1}^{+1} \frac{8\sqrt{1-x^2} + 3\pi(1-x^2)}{10} \int_{-\infty}^{\infty} I(t+\beta x) \cos utdtdx \\
&= \frac{5}{4\pi} \int_{-1}^{+1} \frac{8\sqrt{1-x^2} + 3\pi(1-x^2)}{10} \int_{-\infty}^{\infty} I(t) \cos u(t-\beta x) dtdx \\
&= \frac{5}{4\pi} \int_{-\infty}^{\infty} \int_{-1}^{+1} I(t) \cos ut \cos u\beta x \left\{ \frac{8\sqrt{1-x^2} + 3\pi(1-x^2)}{10} \right\} dxdt \\
&= \frac{5}{40\pi} \int_{-\infty}^{\infty} I(t) \cos utdt \times \left\{ \frac{8\pi J_1(u\beta)}{u\beta} - \frac{12\pi \cos u\beta}{u^2\beta^2} + \frac{12\pi \sin u\beta}{u^3\beta^3} \right\} \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} I(t) \cos utdt \times \left\{ \frac{2J_1(u\beta)}{u\beta} - \frac{3\cos u\beta}{u^2\beta^2} + \frac{3\sin u\beta}{u^3\beta^3} \right\}.
\end{aligned} \tag{5.13}$$

Now the expression between the brackets { } in (5.13) has its first four zeros at

$$u\beta = \begin{cases} 4{\cdot}147 \\ 7{\cdot}303 \\ 10{\cdot}437 \\ 13{\cdot}569 \end{cases} \qquad (5.14)$$

so therefore also must the left-hand side. We can find the zeros of the left-hand side directly from the observations, and search among them (in favourable cases the search is very rapidly concluded) for a set to agree with (5.14) with a given single value of β.

If the line is due to rotation these results must obtain, and conversely we can claim that, if they are found, the probability of rotation being actually the cause of the observed shape is rendered high, the higher the better the agreement and the greater the number of observed lines found to fit.

It will be seen in what follows that in certain circumstances (namely, when the distortion of the line is great) a good approximate value of β may be found otherwise, by merely assuming the original line to be nowhere negative, and if this value be used for a check of (5.14) and agreement found we have strengthened the case for rotational distortion as the cause of the observed shape of the line.

Numerical examples are given below after the next section.

It is to be observed that the position of the zeros in (5.14) depends on the law of darkening; for no darkening they are all approximately at values 0·3 smaller. This is not beyond the bounds of observational test.

5.2. *Other Methods of Finding the Rotational Speed* (Special Cases).—It is important to note in the first place that the magnitude of β in itself is not the deciding factor in the form of the solutions of the equations, but its magnitude in relation to the size and shape of the line considered. Thus the effect (apart from a scale factor) due to β on $I(\zeta)$ is the same as that due to $m\beta$ on $I(m\zeta)$; a broad line requires much higher rotational speed to produce the same distortion in it than does a narrow one. We shall later on divide our discussion into two parts: (*a*) when there is little distortion of the original line, and (*b*) when the distortion is great. The results for the extreme cases can be developed to overlap somewhat and cover the whole range.

In Elvey's * discussion of this problem there is a quotation from my first paper that is rather misleading in view of the point just mentioned. In my first paper I worked out the effect of rotation on a line of form $I(\zeta) = e^{-\alpha^2\zeta^2}$ for various values of β, within the range producing comparatively small distortion. For illustrative purposes a line at 5000 A.U. was chosen, a value 10^4 assigned to α, and the distorted curve $O(\zeta)$ worked out for $\beta = 10^{-4}$ and $10^{-4}\sqrt{2}$. In a footnote to the illustrative example I remarked that the formula ceased to be valid near a velocity of 65 km./sec., *i.e.* $\beta c = 65$ km./sec. This is the remark quoted, but I wish to point out

* *Ap. J.*, **71**, 222, 1930.

that the remark is apropos the particular line chosen and not of general application.

The method followed in that discussion is effectively Method III of this paper and is valid so long as the Taylor expansion is valid. This may be allowable for any value of β, depending on α, *i.e.* on the width of the original line, as is pointed out on p. 555 of the paper; thus had I chosen a line at 5000 A.U. with $\alpha = 10^{-3}$ the limiting value of V would have been 650 km./sec.

If the width of the original line be $2\zeta_0$, i.e. $I(\zeta) = 0$, when $|\zeta| \geqslant \zeta_0$, we divide the discussion into

(*a*) $\beta << \zeta_0$, *i.e.* small distortion. (The velocity may not be small; if ζ_0 is large β may be large also.)

(*b*) $\beta >> \zeta_0$, *i.e.* great distortion. (If ζ_0 is small this may imply quite low velocities.)

Clearly, it can happen that in the spectrum of any given star the effects of rotation may vary widely from one line to another, depending on the form of the original line.

5.3. *Small Distortion.*—Equation 1 (*a*) allows (see § 3, p. 486) in this case a solution of the form

$$I(\zeta) = O(\zeta) - \tfrac{9}{80}\beta^2 O_2(\zeta) + \cdots \qquad (5.31)$$

Near the wings of the line, when $O_4(\zeta)$ and higher differential coefficients may be neglected, we have only the first two terms of the right-hand side of (5.31), and if we assume that *at the wings of the observed line the original line is negligible*, we find

$$O_2(\zeta) = \frac{80}{9\beta^2} O(\zeta),$$

that is to say

$$O(\zeta) = Ce^{-\frac{\zeta}{\beta}\sqrt{\frac{80}{9}}} \qquad (5.32)$$

Thus the wings of the observed line may be roughly exponential in form. This cannot be strictly true as it is inconsistent with the more accurate expression (3.93) or (3.94), but it should yield a fair approximation.

It is useless as a means of finding β as a rule, since there is, first of all, no guarantee that the form is due to rotation, and, secondly, a few trials convince one that the logarithmic differential of $O(\zeta)$ is not nearly constant at the wings. It is often constant over a fair range, but it is hard to justify the selection of this value rather than any later one.

The result (5.32) is perhaps of some interest in providing a possible explanation for an observation of Elvey's * that the profiles of observed lines likely to be deformed by rotation are exponential in form, but his result is obtained by graphical plotting of logarithms of the observed intensities and cannot be taken as establishing anything more than a very rough linearity of the logarithmic derivate of $O(\zeta)$.

In any event if it be shown that a given observed line be strictly exponential

* *Ap. J.*, 70, 141, 1929.

in form, then from (2.2) and its solution it follows that so must the original line be, and the explanation is still to be sought. I am merely pointing out a possible reason for Elvey's approximate result, rather stronger than the well-known fact that nearly any observations when plotted logarithmically are approximately linear.

However, the slope of the observed line at the wing may yield a value of β. Thus, if the observed line can be represented by $Ae^{-a\zeta}$ at the wings, then

$$\beta = \frac{1}{a}\sqrt{\frac{80}{9}}. \tag{5·33}$$

In practice it is not very easy to determine a in the wings of the line, where the observations are few and not very accurate, and the following method has been found more satisfactory than a direct attempt to find a by plotting $\log O(\zeta)$ against ζ.

Near the centre $O(\zeta)$ must be of roughly parabolic shape, and near the wings must tend to exponential form under these circumstances. If $O(\zeta)$ be symmetrical, as it commonly is within experimental error or as a first approximation, then a curve having the required properties is

$$O(\zeta) = O_0 e^{b - \sqrt{b^2 + a^2\zeta^2}},$$

since when ζ is small $\left(\frac{a\zeta}{b} << 1\right)$ this is

$$O = O_0 e^{-\frac{1}{2}\frac{a^2\zeta^2}{b^2}}$$

and when ζ is large $\left(\frac{a\zeta}{b} >> 1\right)$ it is

$$O = O_0 e^b e^{-a\zeta}.$$

One further point of interest is that we can place some sort of upper limit on the value of β by using the fact that $I(\zeta)$ cannot exceed unity. This upper limit is not of great value when $O(0)$ is small, or rather $O(0)/I(0)$, but this mode of attack is generally unsuitable then; we have assumed the distortion small, so that $I(0)$ cannot much exceed $O(0)$, and if any value I_0 be assigned to $I(0)$, then we must have

$$I_0 \geqslant O(0) - \tfrac{9}{80}\beta^2 O_2(0).$$

It is instructive to examine this method from a somewhat different point of view. Consider for simplicity a star with no darkening factor, then the function $G(\beta x) = \frac{2}{\pi}\int_{-1}^{+1} e^{\beta t x}\sqrt{1 - x^2}\,dt$ is $2\frac{I_1(\beta x)}{\beta x}$ and we obtain our solution by expanding in ascending powers of βx, but this cannot be done if βx exceeds the first zero y_1 of $J_1(z)$ in value, since $\frac{z}{I_1(z)}$ has a pole when $z = \pm iy_r$ and the expansion diverges. Thus it is only when the range of integration in (3.93) is less than y_1/β effectively that the procedure is legitimate, *i.e.* $\frac{\beta x \phi(x)}{2I_1(\beta x)}$

or $\psi(x)$ must be negligible when x is $> \frac{y_1}{\beta}$, or in effect $e^{ux}I(u)$ is negligible when $x \geqslant \frac{y_1}{\beta}$; these restrictions make the method of little value.

5.4. *Great Distortion.*—Naturally when the effect of rotation is great it should be then that the value of β is most easily and accurately determined. We shall now show how this may be effected.

We start with

$$O(\zeta) = \frac{5}{4\pi}\int_{-1}^{+1} I(\zeta+\beta t)\left\{\frac{8\sqrt{1-t^2}+3\pi(1-t^2)}{10}\right\}dt \tag{5.40}$$

and

$$I(\zeta) = 0, \ |\ \zeta\ | \geqslant \zeta_0 > 0 \quad \text{where} \quad \beta > \zeta_0 ;$$

we can rewrite (5.40) as

$$O(\zeta) = \frac{5}{4\pi\beta}\int_{-\beta}^{\beta} I(\zeta+u)\left\{\frac{8\sqrt{1-u^2/\beta^2}+3\pi(1-u^2/\beta^2)}{10}\right\}du, \tag{5.41}$$

and since $u < \beta$ we can expand $\sqrt{1-u^2/\beta^2}$, obtaining for $\zeta = 0$ (the central value)

$$h = O(0) = \frac{5}{4\pi\beta}\int_{-\beta}^{\beta} I(u)\left\{\frac{8+3\pi}{10} - \frac{4+3\pi}{10}\frac{u^2}{\beta^2} \cdots\right\}du \tag{5.42}$$

$$= \frac{8+3\pi}{4\pi}\frac{A}{\beta} - \frac{4+3\pi}{8\pi\beta^3}\int_{-\beta}^{\beta} u^2 I(u)du + \cdots \tag{5.43}$$

since $\beta > \zeta_0$, where A is the area of $I(\zeta)$ or $O(\zeta)$, and h and A are known directly from the observation. So we have at once the following approximations for β:—

1st Approximation:

$$\beta_1 = \frac{A}{h}\cdot\frac{3\pi+8}{8\pi}. \tag{5.44}$$

2nd Approximation:

$$h = \frac{A}{h}\cdot\frac{3\pi+8}{8\pi} - \frac{4+3\pi}{8\pi\beta^3}\int_{-\infty}^{+\infty} u^2 I(u)du. \tag{5.45}$$

We have seen in § 4 how this may in turn be found in terms of $O(\zeta)$.

Returning to (5.41) we see that the integrand is the product of $I(\zeta+\beta t)$ and $\frac{8\sqrt{1-t^2}+3\pi(1-t^2)}{10}$ and, referring to fig. 1, the effect of altering ζ will be small until $\zeta+\zeta_0 = \beta$, thus the central part of the observed profile will be broad and flat, *i.e.* we are considering the *Dish-shaped Profiles.* Analytically the same result is obtained on expanding $I(\zeta+u)$ in ascending powers, or better by differentiating (5.41) twice with respect to ζ and integrating by parts, when we find (5.42) and the new result that $O(\zeta)$ and $O''(\zeta)$ are nearly constant when ζ is small. So that *for the Dish-shaped Profiles a good approximation to β is given by* (5.44).

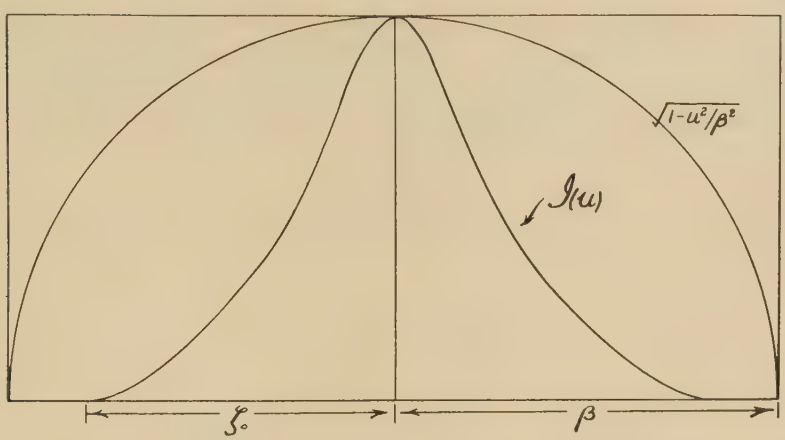

FIG. 1.—*The generation of a distorted line. The two curves,* $I(\zeta+u)_{\zeta=0}$, *and* $\sqrt{1-u^2/\beta^2}$, *represent the genesis of* $O(o)$ *in a star with no darkening.* $O(\zeta)$ *is obtained by displacing* $\sqrt{1-u^2/\beta^2}$ *by* ζ, *multiplying the two curves together and integrating.*

We shall see (§ 6) that (5.44) gives a useful value in less extreme cases.

An Upper Limit for β.—We have by (5.41), since $\sqrt{1-\frac{u^2}{\beta^2}} < 1$,

$$O(\zeta) < \frac{2}{\pi\beta}\int_{-\beta}^{\beta} I(\zeta+u)du,$$

and since $I(\zeta) > 0$ (all ζ), the right-hand side of this expression is

$$< \int_{-\infty}^{\infty} I(\zeta)d\zeta.$$

Thus

$$h = O(o) < \frac{A}{\beta}\cdot\frac{3\pi+8}{8\pi},$$

or

$$\beta < \frac{A}{h}\cdot\frac{3\pi+8}{8\pi}. \qquad (5.46)$$

This is then an upper limit for β, whether the distortion be great or small. The previous equations provide for closer delimitation of β if required.

This result is of interest in that it provides a value of β for use in conjunction with (5.14), and while, where the distortion is large and the value of β found by (5.44) and (5.45) is a good approximation, this inequality is not specially needed, it is quite useful in other cases.

We can now see that the method used by Struve and Elvey is likely to give good approximate values of β, providing the distortion is fairly large, since under these circumstances the form of $O(\zeta)$ does not depend markedly on the form of $I(\zeta)$, but primarily on its area, *i.e.* if the area of the assumed $I(\zeta)$ is correct h is determined mainly by that, and we can regard their method as a similar but better approximation than (5.44) or (5.45).

6. *Numerical Tests and Illustrations.*—In order to examine the relative

accuracy and suitability of various methods described, and to illustrate the application of these results in practice, I have computed from an arbitrary original line the "observed" shapes corresponding to two different rotational speeds, one producing moderate or small distortion, the other producing great distortion. These two "observed" curves are used in illustration; the "calculated" values of β and the original obtained from them can then be compared with the known values.

To simplify the labour of the numerical work a star with no darkening factor has been used. This makes no difference to the *modus operandi*;

TABLE I

The Original Line

$I(\zeta)$

The interval in the argument (ζ) between successive entries in the table is $\frac{1}{12} \times 10^{-4}$

$\zeta \times 10^4$	$I(\zeta)$	$\zeta \times 10^4$	$I(\zeta)$	$\zeta \times 10^4$	$I(\zeta)$	$\zeta \times 10^4$	$I(\zeta)$
0·0	10000	2·5	5241	5·0	1991	7·5	282
	9987		5080		1919		252
	9952		4927		1846		222
	9897		4770		1780		194
	9827		4624		1704		172
	9741		4476		1636		153
0·5	9637	3·0	4342	5·5	1566	8·0	134
	9519		4204		1503		121
	9385		4078		1437		105
	9246		3958		1365		93
	9098		3838		1304		83
	8940		3729		1239		73
1·0	8770	3·5	3618	6·0	1174	8·5	63
	8574		3516		1118		54
	8352		3412		1062		44
	8138		3307		1005		38
	7923		3201		953		34
	7722		3104		897		28
1·5	7513	4·0	3004	6·5	842	9·0	22
	7306		2900		784		18
	7080		2810		733		14
	6879		2724		685		10
	6675		2635		633		7
	6496		2548		582		3
2·0	6310	4·5	2462	7·0	534	9·5	0
	6118		2382		485		
	5928		2300		441		
	5746		2218		398		
	5572		2144		355		
	5408		2065		315		
2·5	5241	5·0	1991	7·5	282		

$$A = \int_{-\infty}^{\infty} I(\zeta) d\zeta = 6{\cdot}270$$

and the appropriate formulæ, where they differ from those already obtained will be stated. They differ little in effect from the formulæ, taking darkening into account, and may be used instead for approximate calculations. I am specially indebted to Miss L. J. Ingram, M.A., for help in this part of the work; without her painstaking and very accurate assistance the appearance of this paper would have been greatly delayed owing to the mere length of the numerical work, a large part of which does not appear in the final results at all, especially the thorough tests on actual stars of various methods.

TABLE II

Moderate Distortion

The "Observed" form $O(\zeta)$ of the line produced from $I(\zeta)$, Table I, by a rotational speed of 100 km./sec., or $\beta = \frac{1}{3} \times 10^{-4}$

$\zeta \times 10^4$	$O(\zeta)$	$\zeta \times 10^4$	$O(\zeta)$	$\zeta \times 10^4$	$O(\zeta)$
0·00	7633	4·25	3178	8·50	297
0·25	7612	4·50	2872	8·75	238
0·50	7549	4·75	2587	9·00	188
0·75	7443	5·00	2323	9·25	146
1·00	7296	5·25	2079	9·50	110
1·25	7109	5·50	1854	9·75	81
1·50	6884	5·75	1648	10·00	58
1·75	6624	6·00	1459	10·25	40
2·00	6331	6·25	1286	10·50	27
2·25	6010	6·50	1128	10·75	18
2·50	5667	6·75	983	11·00	12
2·75	5309	7·00	851	11·25	7
3·00	4941	7·25	732	11·50	4
3·25	4569	7·50	624	11·75	2
3·50	4205	7·75	527	12·00	1
3·75	3848	8·00	441	12·25	0
4·00	3504	8·25	364	12·50	0
4·25	3178	8·50	297		

$$A = \int_{-\infty}^{\infty} O(\zeta) d\zeta = 6{\cdot}244$$

I am also indebted to her for careful checking of the whole of the equations in this paper.

The line used as an original, $I(\zeta)$, has a width of about 2×10^{-3} in ζ, *i.e.* about 10 A., if centred at 5000 A.; the two rotational speeds used are 100 km./sec., *i.e.* $\beta = \frac{1 \cdot 0}{3} \times 10^{-4}$, and 300 km./sec., *i.e.* $\beta = 10 \times 10^{-4}$.

It is an obvious convenience to work the numerical calculations with 10^{-4} as a unit, so that β is regarded as $\frac{1 \cdot 0}{3}$, or 10, and the line as of width 20.

Tables I, II and III give the values of $I(\zeta)$ and $O(\zeta)$ for these lines, and fig. 2 shows them plotted all on the same scale. It is worth remarking that the striking difference between the two curves of $O(\zeta)$ for $V = 100$ km./sec. and $V = 300$ km./sec. is not so apparent in practice, since in the absence of any knowledge of the true value of $I(\zeta)$, especially of $I(0)$, the correct scale is unknown, and $O(\zeta)_{\beta = \frac{1 \cdot 0}{3}}$ is not very different to the eye from $O(\zeta)_{\beta = 10}$

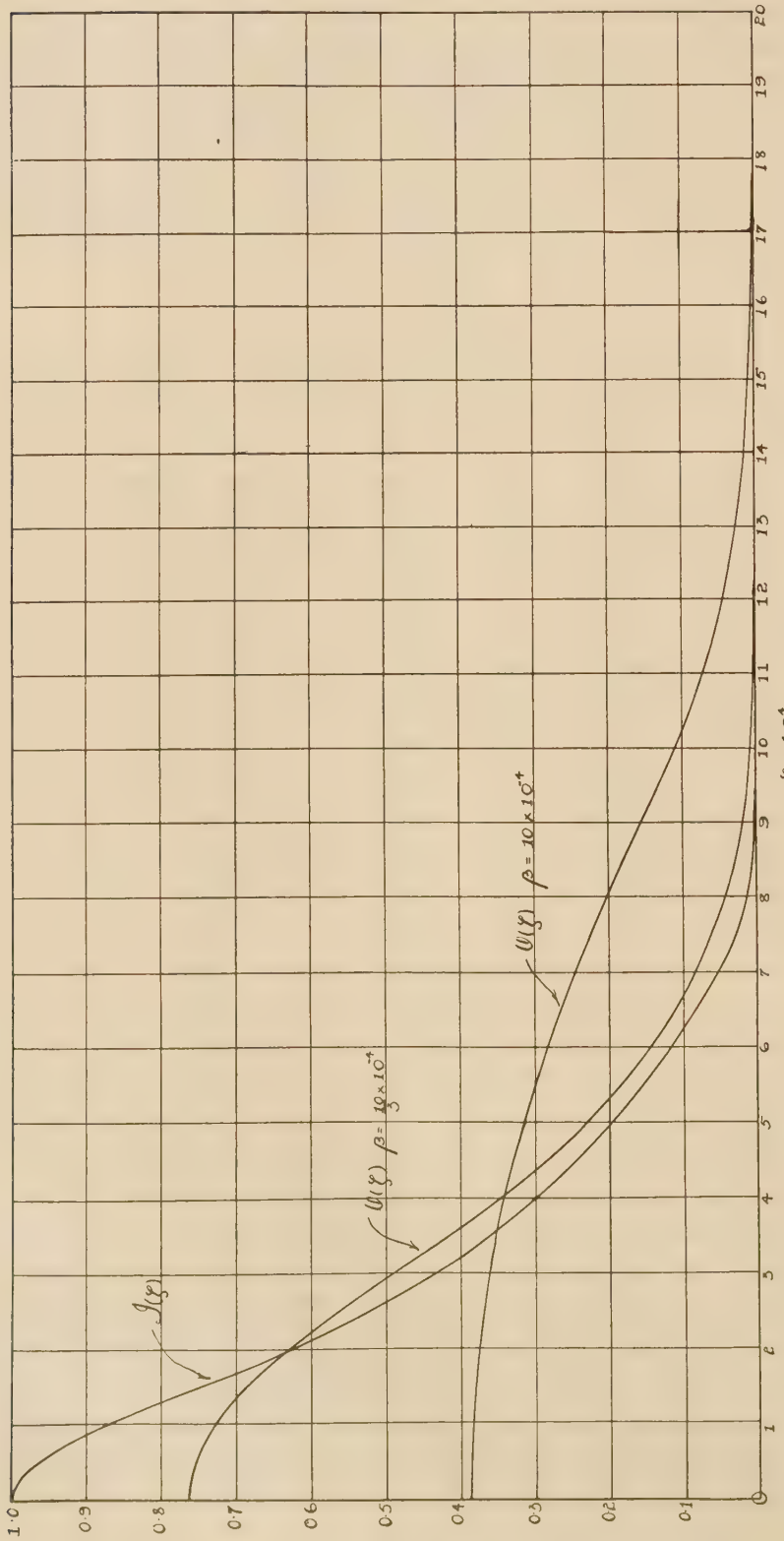

FIG. 2.—*The original line* I(ζ), *and its "observed" distorted forms* O(ζ) *corresponding to rotational speeds of* 100 *and* 300 *km./sec., used for tests of the methods developed.*

when the former is plotted on half the scale in height and double the scale in width, as may be seen on comparing figs. 3 and 4.

Finally, the figures used correspond closely to actual cases, since, although the values of V chosen sound very high, the line is rather wide, and by the rule mentioned on p. 491 the results apply equally to a line like $I(\zeta)$ but plotted on, say, half the scale of ζ, *i.e.* of width 5 A.U., if the velocities, and so β, be halved also.

TABLE III

The "Observed" form $O(\zeta)$ of the line produced from $I(\zeta)$, Table I, by a rotational speed of $V=300$ km./sec., $\beta=10^{-4}$

$\zeta\times10^4$	$O(\zeta)$	$\zeta\times10^4$	$O(\zeta)$	$\zeta\times10^4$	$O(\zeta)$
0·0	3806	7·0	2459	14·0	147
0·5	3800	7·5	2252	14·5	103
1·0	3782	8·0	2030	15·0	69
1·5	3752	8·5	1800	15·5	44
2·0	3708	9·0	1562	16·0	26
2·5	3651	9·5	1329	16·5	14
3·0	3580	10·0	1106	17·0	6
3·5	3493	10·5	905	17·5	3
4·0	3391	11·0	728	18·0	1
4·5	3273	11·5	579	18·5	0
5·0	3140	12·0	455	19·0	0
5·5	2993	12·5	353	19·5	0
6·0	2830	13·0	270	20·0	0
6·5	2652	13·5	202		
7·0	2459	14·0	147		

$$A=\int_{-\infty}^{\infty}O(\zeta)d\zeta=6\cdot239$$

6.1. *The Test for Rotation, and the Determination of the Speed.*—Before applying the test for rotation it is useful to find the upper limit to β, and incidentally to see the degree of approximation of formulæ.

In this case (no darkening) they become

$$h<\frac{2A}{\pi\beta} \tag{6.11}$$

and

$$h<\frac{2A}{\pi\beta}-\frac{1}{\pi\beta^3}\int_{-\beta}^{\beta}u^2I(u)du$$

$$=\frac{2A}{\pi\beta}-\frac{2}{\pi}\left(\frac{2}{\beta}\right)^{\frac{3}{2}}\int_0^{\infty}p^{\frac{1}{2}}O(p+\beta)dp. \tag{6.12}$$

Taking first the case of great distortion, when these formulæ should yield a good approximation (*i.e.* we can replace $<$ by $=$), we get, using Table III, from (6.11)

$$\beta=10\cdot42\times10^{-4}$$

and from (6.12)

$$\beta=10\cdot002\times10^{-4}$$

or $V = 300{\cdot}1$ km./sec., a very close approximation. From Table II, the case of small distortion, we get from (6.11) $\beta = 5{\cdot}21 \times 10^{-4}$, rather high compared with the correct value $3{\cdot}33 \times 10^{-4}$, and on using (6.12) we find little or no change. This is to be expected since the formulæ can hardly apply, except as an upper limit, since 2β is of the same magnitude as the width of $I(\zeta)$ or smaller, and so $\int_{-\beta}^{\beta} I(\zeta)d\zeta$ cannot be taken as A the area of $I(\zeta)$.

Turning now to the test for rotational distortion, by (5.12) and (2.1), the zeros are now simply at the zeros of $J_1(u\beta)$, *i.e.*

$$u\beta = \begin{cases} 3{\cdot}832 \\ 7{\cdot}016 \\ 10{\cdot}174 \\ 13{\cdot}324. \end{cases} \tag{6.13}$$

For the line showing great distortion we find zeros of $\int_{-\infty}^{\infty} O(t) \cos utdt$ at

$$u = \begin{cases} 0{\cdot}382 \\ 0{\cdot}702 \\ 1{\cdot}021 \\ 1{\cdot}339 \end{cases} \text{ yielding respectively } \beta \times 10^4 = \begin{cases} 10{\cdot}01 \\ 9{\cdot}99 \\ 9{\cdot}99 \\ 9{\cdot}97 \end{cases}$$

in excellent agreement with the true value $\beta = 10 \times 10^{-4}$.

The integral is small when u is large, and it is only in cases of great distortion that one can expect to press the results as far as the fourth zero.

Fig. 3 shows the graph of $O(\zeta)$ and of $\int_{-\infty}^{\infty} O(t) \cos utdt$. It is noteworthy that there is no confusion caused by "natural" zeros, *i.e.* ones not due to the rotational element of the shape of $O(\zeta)$.

In the other example shown in fig. 4, small distortion, from Table II, we get by (6.12) zeros at

$$u\beta = \begin{cases} 1{\cdot}13 \\ 2{\cdot}21 \\ 2{\cdot}77 \end{cases} \text{ yielding respectively } \beta \times 10^4 = \begin{cases} 3{\cdot}38 \\ 3{\cdot}17 \\ 3{\cdot}67. \end{cases}$$

The agreement with the correct value of β ($3{\cdot}33$) is good. It could, of course, be made much better by more elaborate numerical integration, but it is more instructive to take the figures as they stand for comparison with the results for $\beta = 10$. We see that, first of all, rather less accuracy is to be expected, and, secondly, owing to the much larger values of u involved, at most three zeros can be picked up with certainty. Again no trouble is experienced with "natural" zeros, and we may anticipate little trouble of this kind with actual profiles.

A word of warning about the evaluation of $\int_{-\infty}^{\infty} O(t) \cos utdt$ may be welcome. In numerical work this integral is not in fact evaluated, but

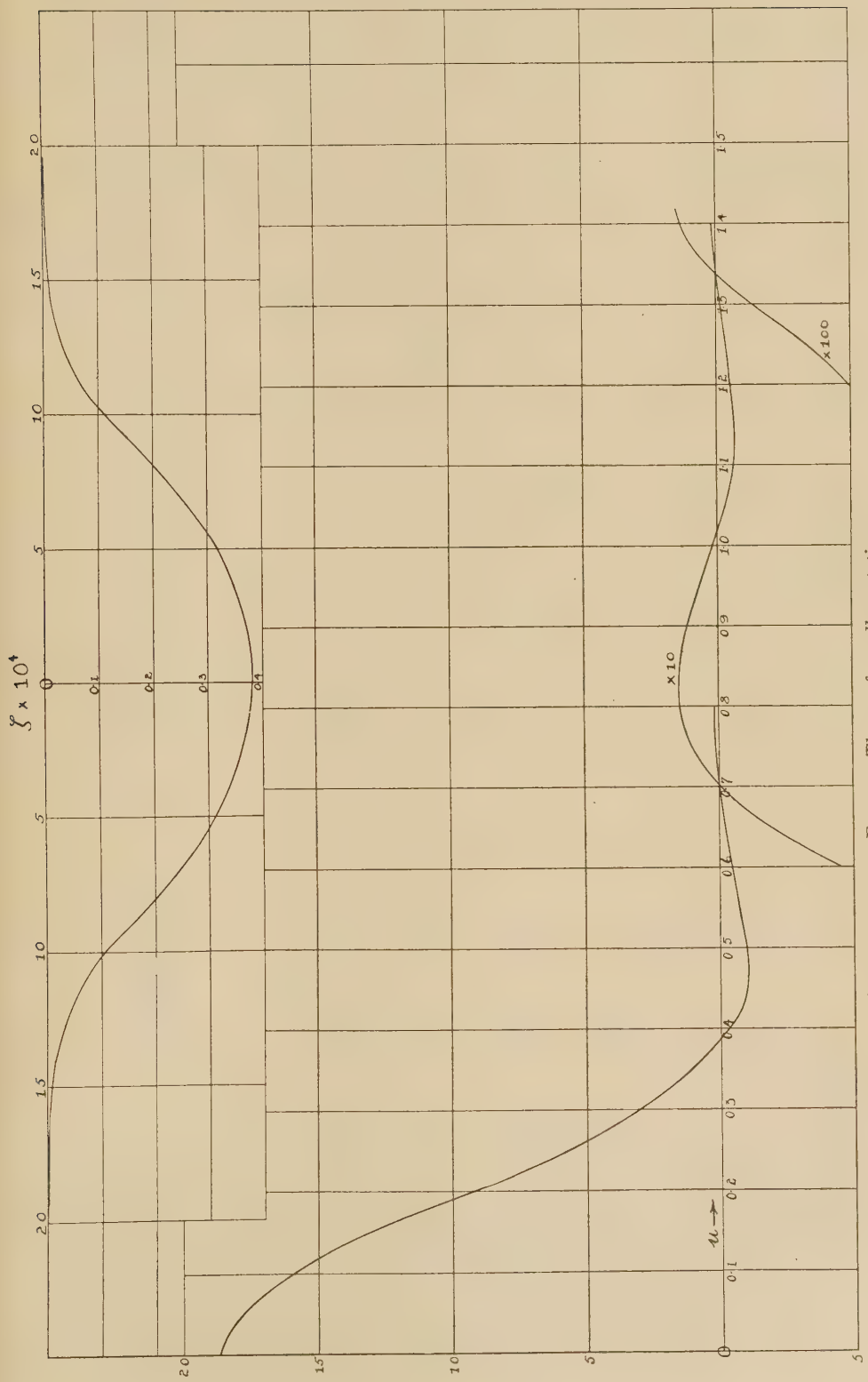

FIG. 3.—*The test for stellar rotation.*

The observed form $O(\xi)$ *and* $\int_{-\infty}^{\infty} O(t) \cos ut\,dt$ *showing zeros of the latter, when* V = 300 *km./sec. (Great distortion.)*

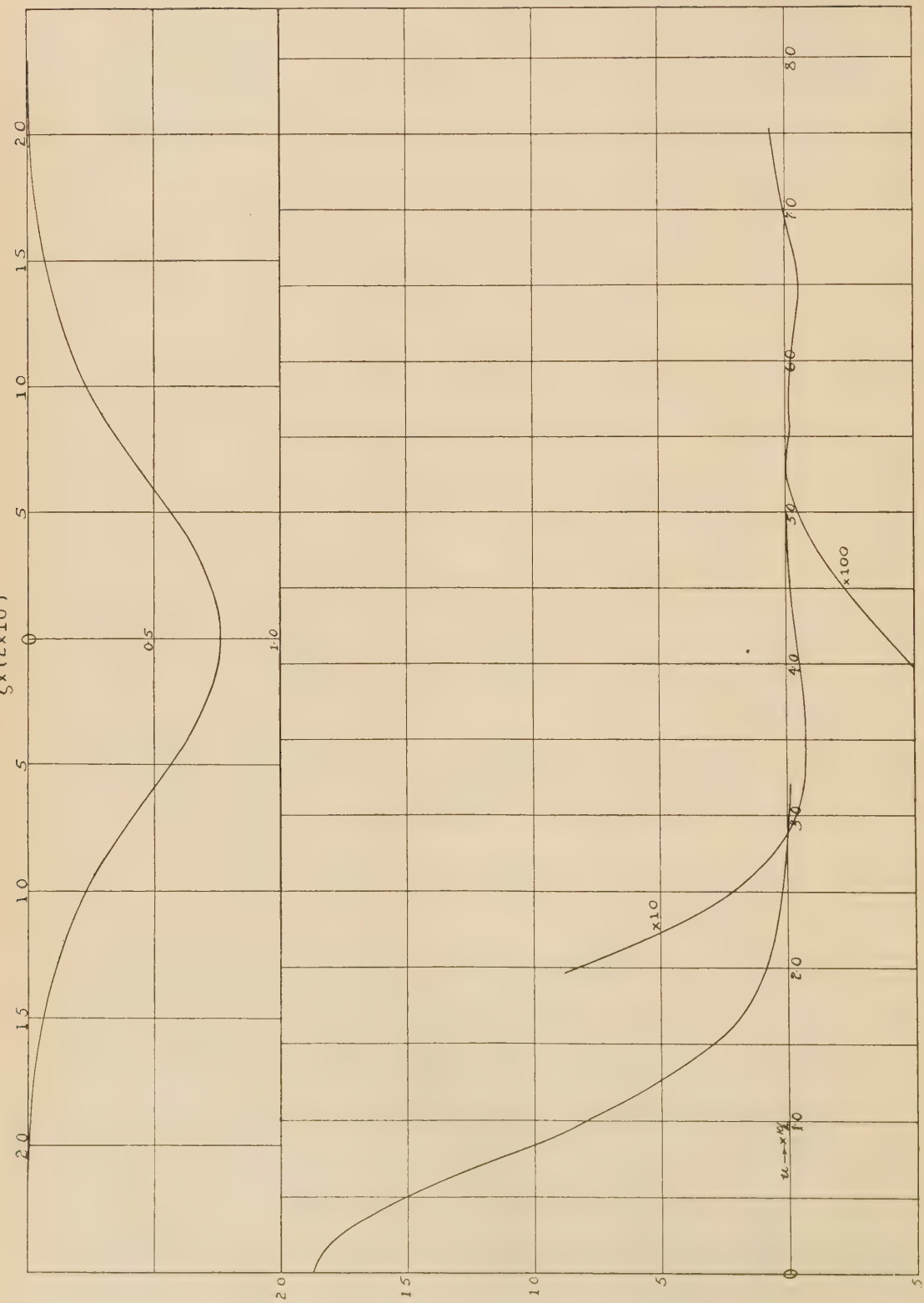

FIG. 4.—*The test for stellar rotation.*

use is made of some formula of mechanical integration, Simpson's rule, Weddle's rule or Gregory's formula being suitable and convenient. Thus in fact we evaluate $\sum_{r=-n}^{+n} O(rw) \cos urw \,.\, w$ where w is the interval between successive ordinates, and $O(rw) = O(-rw) = 0$ if $r \geqslant n$, but this procedure necessarily yields a periodic function with period, in u of $2\pi/w$, whereas the integral is not periodic. It is therefore necessary to choose w so small that $\frac{2\pi}{w}$ is considerably larger than twice the last zero to be found, otherwise the value obtained may be considerably in error.

To sum up, we see that the analysis of line shape gives definite and accurate results, and the approximate formulæ for β are accurate when the distortion is great; when it is small the value of β found is considerably too large. In any case then we must reject values of β inconsistent with the upper limit calculable by (6.11) or (6.12).

With sufficiently good determinations of $O(\zeta)$ for lines showing great distortion it should be possible to test whether the results agree more closely with (6.13) or (5.14), *i.e.* we can gain some idea of the law of darkening. It is doubtful whether present measurements are sufficiently accurate for the purpose, but the point is discussed in the following paper.

6.2. *The Calculation of the True Form of the Line.*—We will take first of all the case of great distortion, $\beta = 10$ (Table III).

It is at once clear that (5.31) is quite inapplicable; such an expansion is invalid, as will be seen below, where Method I of § 3 is applied.

The asymptotic expansion here works to perfection. The formula for no darkening, corresponding to (4.4), is

$$I(u) =$$

1st Term:

$$-4\left(\frac{\beta}{2}\right)^{\frac{3}{2}} \int_0^\infty p^{\frac{1}{2}} O_3(p+u+\beta)dp\,;$$

2nd Term:

$$\frac{3}{4}\left(\frac{\beta}{2}\right)^{\frac{1}{2}} \int_0^\infty p^{\frac{1}{2}} O_2(p+u+\beta)dp\,;$$

3rd Term:

$$-\frac{33}{128}\left(\frac{2}{\beta}\right)^{\frac{1}{2}} \int_0^\infty p^{\frac{1}{2}} O_1(p+u+\beta)dp\,;$$

4th Term:

$$\frac{249}{2048}\left(\frac{2}{\beta}\right)^{\frac{3}{2}} \int_0^\infty p^{\frac{1}{2}} O(p+u+\beta)dp, \qquad (6.21)$$

where from Table III we find the original as given in Table IV. The agreement is practically perfect, and, on the scale of fig. 2, the calculated and true values of $I(\zeta)$ are indistinguishable. The accuracy far exceeds the accuracy to which the observational data are known. The method of fitting a curve of the form $e^{b-\sqrt{b^2+a^2\zeta^2}}$ cannot be applied, as it is readily demonstrable that no curve of this form will fit $O(\zeta)$ even moderately well.

TABLE IV

Original ($I(\zeta)$) calculated by (6·21) from the "observed" value ($O(\zeta)$) of Table III, for $V = 300$ km./sec., great distortion

	Contributions from			$I(\zeta)$	
ζ	1st Term	2nd Term	3rd Term	Calc.	True
0·0	0·888	0·107	0·016	1·011	1·000
0·5	0·867	0·090	0·010	0·970	0·963
1·0	0·789	0·074	0·006	0·873	0·878
2·0	0·576	0·049	0·003	0·631	0·632
3·0	0·397	0·032	0·002	0·432	0·434
4·0	0·278	0·018	0·001	0·298	0·300
5·0	0·189	0·010	0·000	0·200	0·200
6·0	0·113	0·004	0·000	0·118	0·118
7·0	0·051	0·001	0·000	0·053	0·053
8·0	0·014	0·000	0·000	0·014	0·013
9·0	0·007	0·000	0·000	0·001	0·001
10·0	0·000	0·000	0·000	0·000	0·000

The remaining method is Method I of § 3. $O(\zeta)$ can be represented with considerable accuracy by

$$O(\zeta) = \frac{O_0}{(1 + \cdot 02025425p^2)(1 - \cdot 0121647p^2 + \cdot 00014645 3p^4)}$$

$$= \frac{337{,}121 O_0}{(p \pm \iota 7\cdot 027)(p + 7\cdot 880 \pm \iota 4\cdot 533)(p - 7\cdot 880 \pm \iota 4\cdot 533)}$$

leading to the integrals from 0 to ∞ of expressions from the complex integral as follows :—

$$I(u) = \int_0^\infty e^{-ut} \frac{10t}{2I_1(10t)} \{3\cdot 588 \sin 7\cdot 027t - 4\cdot 1639 \sinh 7\cdot 880t \cos 4\cdot 533t + 1\cdot 6767 \cosh 7\cdot 880t \sin 4\cdot 533t\} dt. \quad (6.22)$$

The resulting curve for I is shown in fig. 5. It is a fair approximation, running too high in the centre and at the wings, and being too low in between. The inapplicability of (5.31) can be seen from (6.22), where the integrand is not negligible until t is much larger than ·383, or $\frac{1}{10}$ the first zero of $J_1(x)$.

Of course a polynomial of higher degree would yield a better result, but, as can be seen, the labour of the computation becomes excessive, especially as the asymptotic expansion yields accurate results comparatively easily.

The real difficulty to be expected in practice is readily seen. As we saw earlier (p. 495), the shape of the central part of a greatly distorted line does not much depend on the shape of the original line—it is the wings that matter; but this means that to find $I(\zeta)$ accurately we must know their shape well; indeed, as Table IV shows, so well that the third differential coefficient may be well known, since it occurs in the first term of (6.21), and contributes most largely to $I(\zeta)$.

Approximations to the shape will naturally lead to results similar to that given by (6.22), so that little is gained by using specially smoothed data.

Thus we cannot hope to find $I(\zeta)$ at all accurately from such measurements as exist at present if $O(\zeta)$ is much distorted; (6.22) may, however, yield a quite useful rough approximation.

Taking now the case of small distortion, $\beta = \frac{1}{3}^0$ (Table II), we find similar results, which to save space will not be given in detail.

The asymptotic expansion works well but does not converge rapidly, and even with a fifth term added to (6.21) is a few per cent. out. In illustration the value for $I(0)$ only will be quoted; for the later values, the larger u is, the better the agreement with the true value $I(u)$. We find instead of 1·000 for $I(0)$ the value 0·936, about 6 per cent. too low.

An alternative formula for $I(0)$ obtained by differentiating (6.21) with respect to u, and setting $u=0$, gives (since $I'(0)=0$), on substituting from (6.21),

$$I_0 = \frac{3}{2}\left(\frac{\beta}{2}\right)^{\frac{1}{2}}\int_0^\infty p^{\frac{1}{2}}O_2(p+\beta)dp - \frac{3}{16}\left(\frac{2}{\beta}\right)^{\frac{1}{2}}\int_0^\infty p^{\frac{1}{2}}O_1(p+\beta)dp + \text{ etc.}$$

This is useful since $O_3(\zeta)$ is not required, and yields in this case the slightly better approximation 1·020 for $I(0)$, *i.e.* only about 2 per cent. in error.

A simple test for the applicability of the asymptotic expansion is obtained by integrating (6.21) with respect to u from 0 to ∞, thus obtaining $A/2$, the point being that this quantity is always known in practice whereas $I(0)$ is not (and though it may be known not to exceed some given value, this as a rule is much too great an upper limit to be useful). Thus in this case we find the following results (Table V), with the given values of β. It is clear that the series converges very slowly, but four terms seem to be about sufficient for practical purposes, *i.e.* accuracy to 5 per cent. or so.

TABLE V

Contributions from the Different Terms of the Asymptotic Expansion to $A/2$

$\beta \times 10^4$	3·0	3·5	4·0
1st term	1·885	2·123	2·248
2nd ,,	·622	·550	·474
3rd ,,	·283	·203	·145
4th ,,	·150	·086	·051
Total	2·941	2·962	2·918

$$A/2 = 3\cdot122$$

It must be remembered that this is a fairly severe case for this method. An attempt to apply Method I is abortive, the difficulty being that the integrands in the corresponding expression to (6.22) oscillate violently over a large range. The integration can of course be performed and probably the result would be satisfactory, since the first term only (which converges quickly) gives $I(0)$, only about 10 per cent. too large, and the other two terms should yield a small negative contribution, but as a practical method for small distortion this one seems useless.

It is clear that the method of simple expansion giving 3·993 cannot apply, and indeed we get, on trying it, only 0·857 for $I(0)$, much too small a value.

There seems to be a real difficulty in computing $I(\zeta)$ from $O(\zeta)$ even when β is known correctly. Theoretically the problem is solved, but the practical use of the results involves approximations of some form or other, *e.g.* approximate functional representation, or the termination at some convenient point of a series, and the practical difficulty arises in the following way. When β is large, or rather when the distortion is great, the computations are

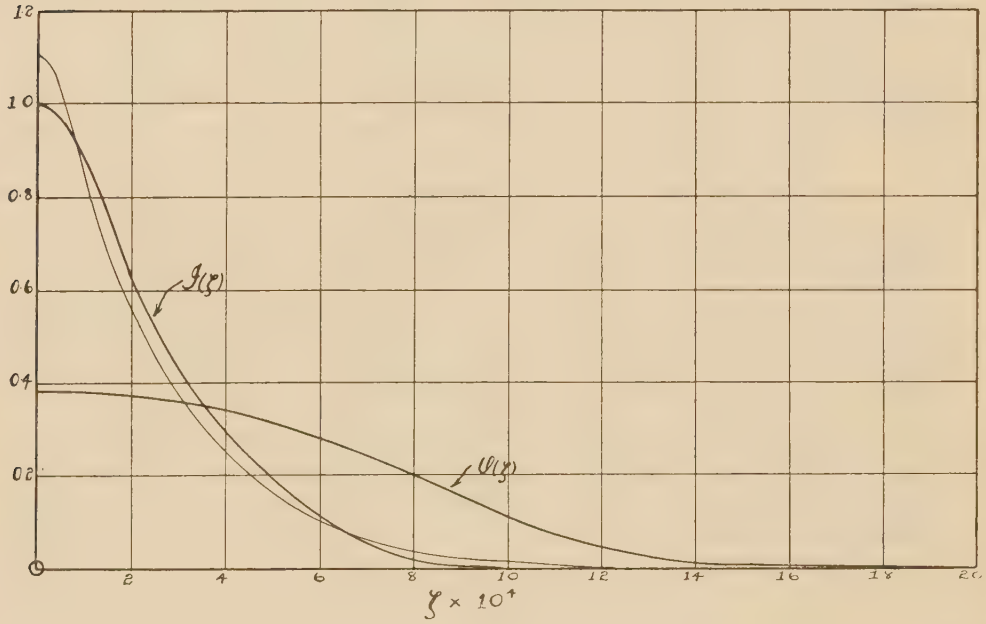

FIG. 5.—*The computation of* I(ζ) *from* O(ζ).

readily carried out and yield accurate results; when the distortion is small, the numerical work is only carried out with difficulty and the approximations are poorer: but unfortunately it is precisely in the former case that the observational material is poor and not able to yield (at present) a good determination of $O(\zeta)$, which we have seen must, for this purpose, be well known *in the wings*. When the observed form of $O(\zeta)$ is likely to be well known (small distortion) we lack a satisfactory calculus. I have some hopes of other lines of attack, but it does not seem wise to hold up the announcement of these results meantime. Clearly, were these properties of the functions of the form $f(a) = \int_0^\infty e^{-ax} \frac{x}{I_1(x)} dx$ known we might expect to use them profitably, but they are obviously transcendents at least more complicated than the logarithmic derivate of the Gamma function, and at present nothing seems known about them, and I have not been able to do more than this paper shows.

Method V here shows its power. It is in effect a modification of

Schwarzschild's method. We here, in this rather difficult case of small distortion, find excellent agreement between the true and computed $I(\zeta)$ as follows :—

		J = 0	1	2	3	4	5	6	7	8	9	10
$I(\zeta)$	True	100	88	63	43	30	20	12	5	1	0	0
	Computed	99	87	63	43	30	19	11	6	2	0	0

It is clear that difficulty will only arise over the uncertainty in the positions of the zeros. Tests on actual stars will decide the utility of the method, but all the indications are favourable.

In conclusion I wish to express my indebtedness to Dr. Paul White for his assistance in discussion and to Dr. L. J. Comrie for kindly supplying tables of $J_1(x)$ at intervals of ·001 in x in the neighbourhood of the zeros of the function in (5.13).

Summary

The effect of rotation of a star on the form of its spectrum lines as observed is discussed. The observed line profile depends on the "true" or original form, on the law of darkening (slightly) and on the star's rotational speed.

The observed form of a line is not entirely arbitrary but owes its shape in part to its mode of production. It is shown how, by a suitable analysis of this observed shape, it can be determined whether it is distorted by stellar rotation, and how, if the star be rotating, its speed can be found directly from the observed line without assumptions concerning the true form of the line.

The integral equation connecting the true and observed profiles is solved and methods of practical use of the solution discussed, whereby the true form can be computed from the observed form and the rotational speed.

Various special cases are discussed, and a simple formula giving an upper limit for the rotational speed is derived.

It is pointed out that high rotational speed in itself is not necessary to produce great distortion of a line; it depends also on the width of the original line. The dish-shaped profiles of Elvey, Shajn and Struve are examined and shown to be characteristic of great distortion.

Numerical illustrations are given, and indications of the accuracy attainable by various methods.

The possibility of testing a law of darkening towards the limb in a rotating star by these methods is pointed out.

THE ROTATIONAL SPEEDS OF THE STARS.

J. A. Carroll, M.A., Ph.D., and L. J. Ingram, M.A.

In this paper the methods developed by one of the writers in another communication to the Society (*M.N.*, 93, 478) * are applied to a few selected stars, with a view to (*a*) discovering whether the effects on spectrum lines supposedly due to stellar rotation do in fact exist, and (*b*) computing the rotational speeds.

The paper is not intended to be exhaustive but merely illustrative, and will be followed later by further results for other stars. We wish to emphasise that, pleasing as it is to the writers to find theoretical methods working in practice, and novel as they are, the interesting results obtained are only possible because the observational material from the labours of Struve and Elvey at the Yerkes Observatory was ready to hand for application of the theory. That the results are so striking and satisfactory must be taken as high testimony to their skill in a very difficult field, since observational errors of much magnitude would make the calculations to be described abortive.

The stars selected have been chosen on the grounds of being (*a*) representative; (*b*) having well-determined line profiles as judged by the "scatter" of the observations about a smooth curve through them; (*c*) having lines showing different grades of rotational distortion.

The method of § 5.1 of Paper I has been applied directly to these stars, and the diagrams show the results.

No attempt has been made to compute the original form of the lines, it is doubtful whether the accuracy attainable warrants it, and it is of subsidiary interest in the present discussion, which is directed to show the reality of the rotational effect and its magnitude.

The usual law of darkening $\frac{2+3\cos\theta}{5}$ has been assumed as in the theoretical work.

In dealing with the observational material the following procedure has been followed.

The observations used were published by Elvey † in the form of rather small-scale reproductions. These were copied directly by contact Playertype process on to lantern slide plates to avoid distortion. These negatives were then placed in a lantern and enlarged until the projected image was on a predetermined scale of intensities, and the observed points marked on squared paper placed on the screen; in this way it is hoped that none of the accuracy of the originals has been lost. A smooth and roughly symmetrical curve was drawn through the observed points and this curve employed to obtain the numerical values used.

* Afterwards referred to as Paper I. † *Ap. J.*, 70, 141, 1929; 71, 221, 1930.

In figs. 2 and 3 the observed points are shown as small circles in the upper part of the diagrams, where the curve used as the line profile is shown.

1. 44 ι *Orionis*.—Observed profiles are available in this star for three lines, *He* 4387, *He* 4471, and *Mg* II, 4481. Two independent determinations of line profile of each of the two helium lines exist; both are wide and not too shallow; one determination only exists for the magnesium line, which is

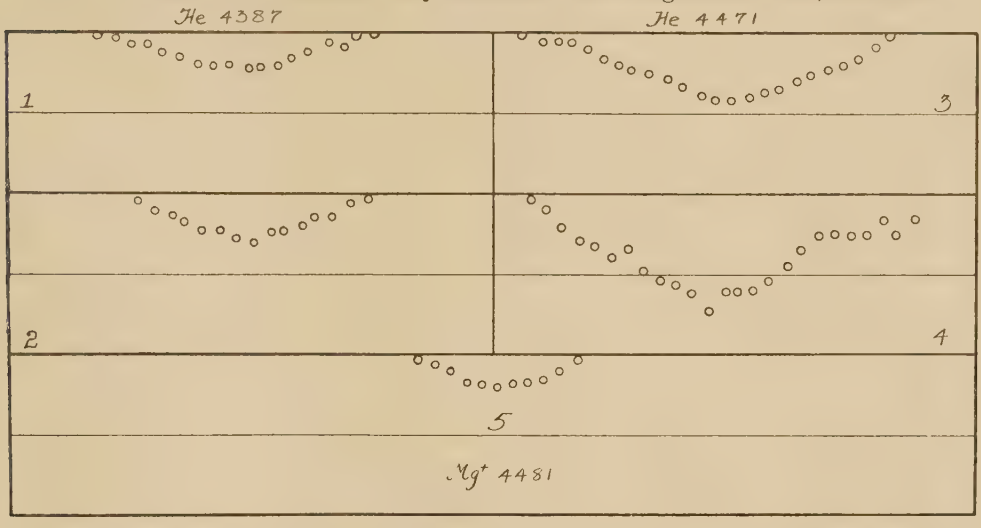

FIG. 1.—He *and* Mg^+ *lines in* 44 ι *Orionis, on the same scale for comparison (after Elvey). The scale of absorption (vertical) is such that the distance between the horizontal lines represents* 20 *per cent. of the continuous background.*

very shallow and rather smaller than the helium lines. All the observations are shown on the same scale in fig. 1. The results obtained from line shape analysis are illustrated in figs. 2 and 3 for *He* 4387 and *He* 4471 respectively. Taking the observations in order as numbered in fig. 1, we have

He 4387.—

1.1. The upper limit formula gives $10^4\beta < 3.9$; the second approximation yields no very definite value for β but is consistent with considerably smaller values.

From the zeros of $\int_{-\infty}^{\infty} O(t) \cos utdt$ we obtain

$$\left.\begin{array}{l} \text{1st zero } 10^4\beta = 3{\cdot}7 \\ \text{2nd ,, } 10^4\beta = 3{\cdot}6 \end{array}\right\} \text{ mean } 3{\cdot}7.$$

This is good agreement and we can say the line shows distinctly the rotational shape; the observational accuracy is hardly sufficient to warrant using the third zero.

Since these values of β are little less than the upper limit, we appear to have here a case of great distortion, but it will easily be seen that even so the original line must be quite wide. The corresponding equatorial line of sight velocities are 111 and 108 km./sec.

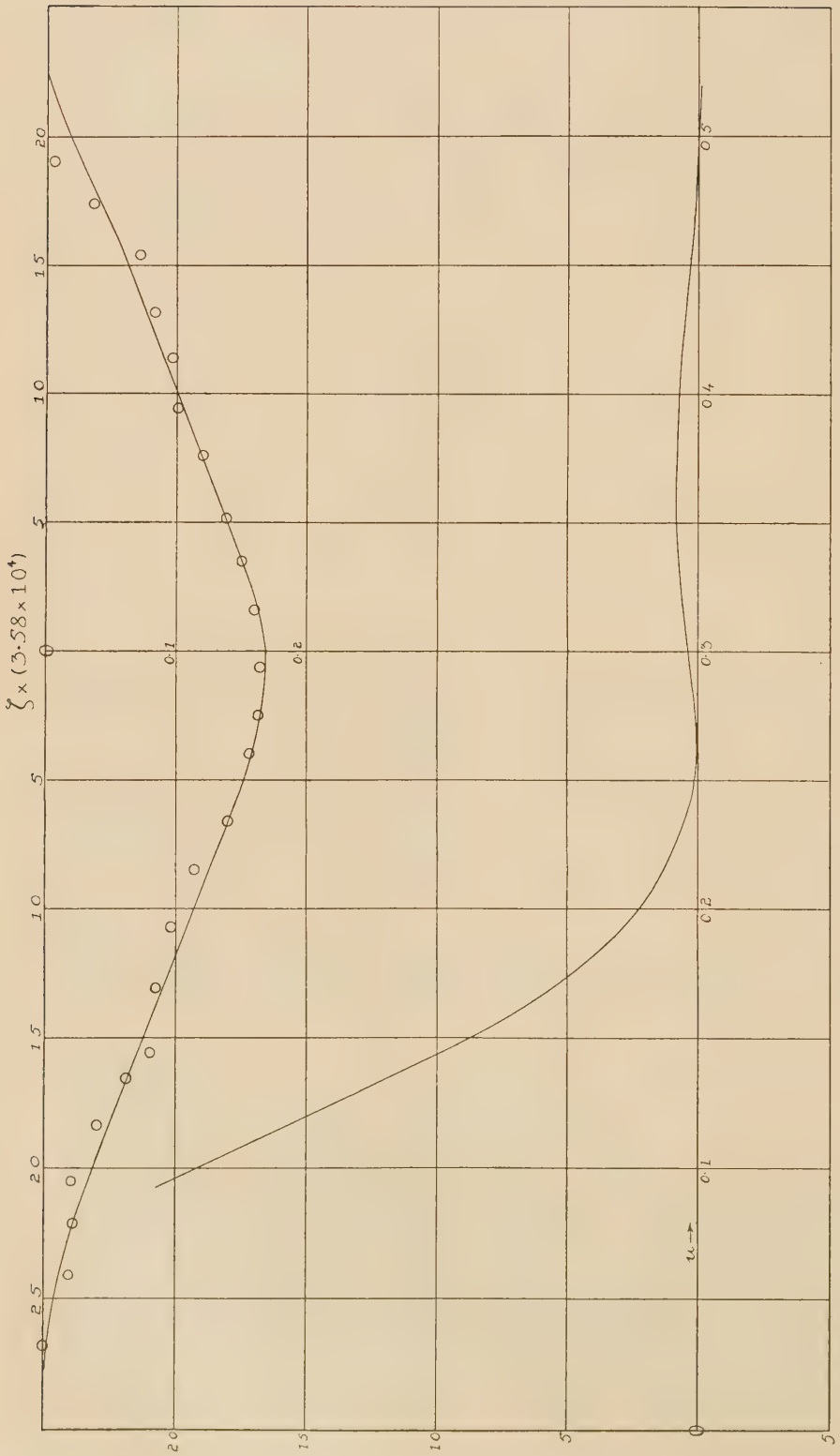

FIG. 2.—*Shape analysis.* 44 *ι Orionis.* He 4387. O(ξ) *and* $\int_{-\infty}^{\infty} O(t) \cos ut\,dt$.

The small circles represent the observed values (Elvey), the continuous line through them the adopted form of O(ξ), *ordinates are absorption as a fraction of continuous background intensity. The lower curve is the integral on an arbitrary scale.*

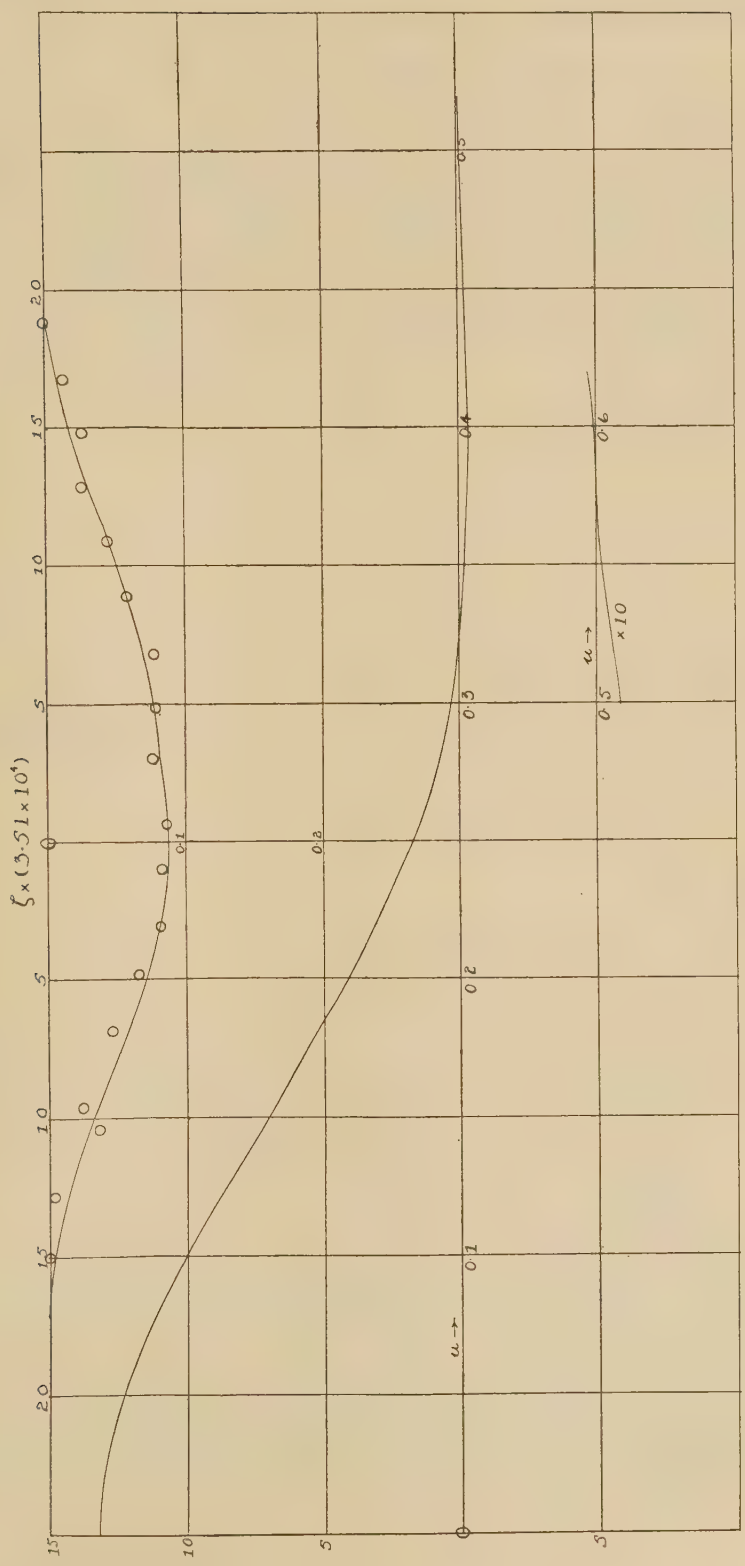

FIG. 3.—*Shape analysis.* 44 *ι Orionis.* He 4471. $O(\xi)$ *and* $\int_{-\infty}^{\infty} O(t) \cos u t dt$.

The small circles represent the observed values (Elvey), the continuous line through them the adopted form of $O(\xi)$, *ordinates are absorption as a fraction of continuous background intensity. The lower curve is the integral on an arbitrary scale.*

1.2. Here we have $10^4\beta < 3{\cdot}7$, and the shape analysis yields

$$\left.\begin{array}{l}\text{from 1st zero } 10^4\beta = 3{\cdot}9\\ \text{,, 2nd ,, } 10^4\beta = 3{\cdot}5\end{array}\right\} \text{mean } 3{\cdot}7.$$

This is good enough agreement to make rotational distortion plausible. The values of β are scarcely different from that obtained from 1.1, and the difference is well within estimated experimental error. The internal consistency of values from *one* line is the rest. Accidental presence of the rotational element of shape will presumably be revealed by wildly discordant values from different lines. The values, too, for any one line are to be judged also by the upper limit for *that* line; this they must not exceed.

From *He* 4471 we have

1.3. $10^4\beta < 4{\cdot}6$.

$$\left.\begin{array}{l}\text{1st zero } 10^4\beta = 4{\cdot}4\\ \text{2nd ,, } 10^4\beta = 4{\cdot}2\end{array}\right\} \text{mean } 4{\cdot}3.$$

1.4. $10^4\beta < 3{\cdot}2$.

$$\left.\begin{array}{l}\text{1st zero } 10^4\beta = 3{\cdot}4\\ \text{2nd ,, } 10^4\beta = 4{\cdot}5\end{array}\right\} \text{mean } 4{\cdot}0.$$

Here (in 1.4) there is definite discordance of the first and second zeros, but as they are both consistent with all the other results as establishing the rotation, it seems proper to include both these values in the determination of β. It will be seen from fig. 1 that this line is irregular in shape, presumably partly blended with other lines, and so too much must not be expected.

Collecting these results we get the values

$$10^4\beta = 3{\cdot}7\ ;\quad 3{\cdot}7\ ;\quad 4{\cdot}3\ ;\quad 4{\cdot}0.$$

Mg$^+$ 4481.—

1.5. $10^4\beta < 3$.

$$\left.\begin{array}{l}\text{1st zero } 10^4\beta = 2{\cdot}7\\ \text{2nd ,, } 10^4\beta = 2{\cdot}5\end{array}\right\} \text{mean } 2{\cdot}6.$$

These values are consistent with rotational distortion, and indeed the line has every appearance of it, but they are very low compared with the other figures from the *He* lines. It is true that this line is smaller and so errors of observation are more important, and assuming, as seems likely, that such errors would on the whole lessen the observed area and lose the wings of the line, the discrepancy is in the direction to be expected, but further observation of this line is much to be desired.

Allowing this observation equal weight with the *He* line values we get as a mean $10^4\beta = 3{\cdot}6$, and this value is out by about three times the probable error, allowing it half weight only we get $10^4\beta = 3{\cdot}7$ and it is out by about twice its new assumed error.

The line *He* 4471 is irregular and on the face of it not a very suitable line for this purpose. The results from it are high compared with the others, and while the value from *Mg*$^+$ 4481 is likely to be too low, that from *He* 4471 is probably too large owing to the inclusion of blends.

Possibly a fair weighting of the results would be $\frac{1}{2}$ each for 4471 and 4481 and 1 for 4378, giving $10^4\beta = 3{\cdot}7$.

Of course one cannot at this stage say much about the accuracy of the results—it is rather surprising to get as good agreement as is found—but the above discussion at any rate may be taken as showing that the observations as a whole are plausibly consistent.

We think this evidence is enough for us to feel reasonably sure that this star is rotating and that its equatorial line of sight velocity is in the neighbourhood of 110 km./sec.

It is of interest to add Struve's and Elvey's estimates of 75 km./sec. and 50 km./sec. respectively from the *Mg* 4481 line; these agree very well with the value (78) obtained by shape analysis from the same curve, but are probably rather low.

2. 44 ζ *Persei*.—This star has lines of much less flattened shape than 44 ι Ori. The same lines have been measured by Elvey, and yield the following results :—

2.1. *He* 4387.—$10^4\beta < 2{\cdot}9$ (two independent profiles used). The zeros occur at (on the scale used)

For the 1st profile,

$$u = \begin{Bmatrix} 0{\cdot}90 \\ 1{\cdot}17 \\ 1{\cdot}70 \end{Bmatrix} \quad \text{corresponding to} \quad 10^4\beta = \begin{Bmatrix} 1{\cdot}31 \\ \dots \\ 1{\cdot}22 \end{Bmatrix} \text{ mean} = 1{\cdot}27.$$

For the 2nd profile,

$$u = \begin{Bmatrix} 0{\cdot}88 \\ 0{\cdot}96 \\ 1{\cdot}47 \\ 1{\cdot}51 \end{Bmatrix} \quad \text{corresponding to} \quad 10^4\beta = \begin{Bmatrix} 1{\cdot}34 \\ 1{\cdot}23 \\ 1{\cdot}42 \\ 1{\cdot}38 \end{Bmatrix}.$$

Clearly the first and last values agree best (with $10^4\bar{\beta} = 1{\cdot}36$). The other zeros are presumably the natural zeros of $I(\zeta)$. The value of β obtained is well below the upper limit of $2{\cdot}9 \times 10^{-4}$ and we have here a case of small distortion.

2.2. *He* 4471.—As this line appears too irregular for good determinations only one profile has been used as a check on 2.1. The zeros are at

$$u = \begin{Bmatrix} 0{\cdot}44 \\ 0{\cdot}89 \\ 1{\cdot}62 \end{Bmatrix} \quad \text{corresponding to} \quad 10^4\beta = \begin{Bmatrix} \dots \\ 1{\cdot}30 \\ 1{\cdot}26 \end{Bmatrix}.$$

The line then shows the rotational effect ; it has an early natural zero and possibly two rotational zeros giving a mean value of $10^4\beta$ of 1·28.

2.3. Mg^+ 4481.—This line is very peaked in form as published by Elvey, and as expected shows no trace of rotational distortion. No zero is found up to values of u far beyond any agreement with 2.1 or 2.2. The line is rather small and perhaps not well determined. Elvey and Struve give from this line an estimate of 50 or 60 km./sec. for the rotational speed, but this must be largely obtained from an estimate of what is needed to

reduce the central intensity they assumed for $I(0)$ to the observed value; the rotational shape is well masked in some way.

Our mean value of β is $1{\cdot}30 \times 10^{-4}$, giving an equatorial line of sight velocity of 39 km./sec.

The *He* lines seem to establish the rotation quite satisfactorily, and, given this, Struve's and Elvey's estimate can be given due weight and is in good agreement with our result when it is remembered that their method is hardly able to detect velocities less than about 50 km./sec. in such a case.

The profiles used are illustrated in fig. 4.

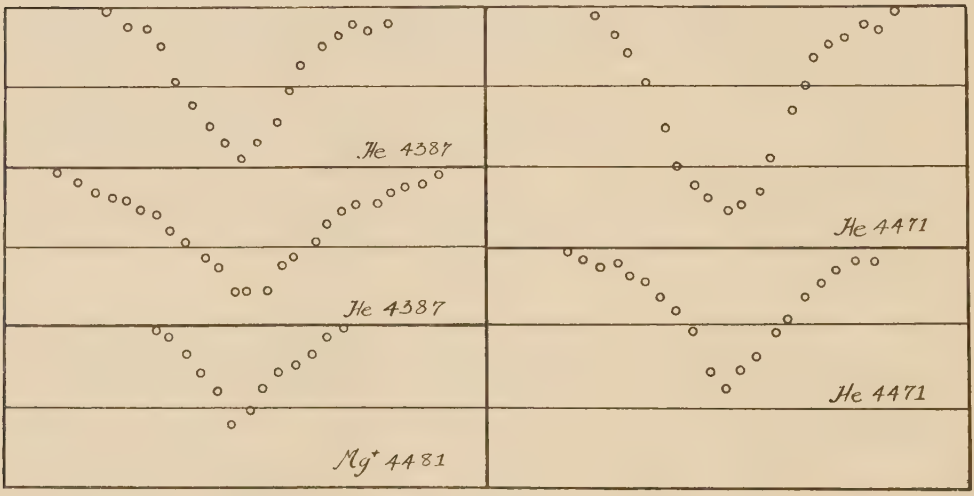

FIG. 4.—44 ζ *Persei.*

The scale of absorption (vertical) is that of fig. 1.

3. 39 λ *Orionis.*—This star has been chosen as having a large Mg^+ 4481 line so that good agreement is to be expected between it and *He* 4387, the two profiles used.

3.1. *He* 4387.—$10^4\beta < 6$.

Three rotational zeros can be picked up with no "natural" zeros. The value of β is near the upper limit, showing therefore great distortion as is to be expected from the form of $O(\zeta)$ and of $\int_{-\infty}^{\infty} O(t) \cos utdt$.

The zeros are at (on the scale used)

$$u = \begin{Bmatrix} \cdot 20 \\ \cdot 33 \\ \cdot 48 \end{Bmatrix} \quad \text{corresponding to} \quad 10^4\beta = \begin{Bmatrix} 5{\cdot}9 \\ 6{\cdot}3 \\ 6{\cdot}2 \end{Bmatrix},$$

giving a mean value of $6{\cdot}1 \times 10^{-4}$ for β.

3.2. Mg^+ 4481.—$10^4\beta < 5{\cdot}9$.

Again three rotational zeros, without natural zeros, can be found with β near the upper limit. The zeros are at

$$u = \left\{\begin{matrix}\cdot 21\\ \cdot 36\\ \cdot 47\end{matrix}\right\} \quad \text{corresponding to} \quad 10^4\beta = \left\{\begin{matrix}5\cdot 6\\ 5\cdot 7\\ 6\cdot 2\end{matrix}\right\},$$

giving a mean value of $5\cdot 8 \times 10^{-4}$ for β.

The agreement between the two lines is excellent, and both lines show the rotational element of shape markedly. The mean value of the rotational speed works out as $\beta = 6\cdot 0 \times 10^{-4}$, or an equatorial line of sight velocity of 180 km./sec., in agreement with Struve's and Elvey's estimate of 200 km./sec.

For this star the rotational speed seems exceptionally well determined, probably well within 10 per cent., and the rotation clearly established.

It is of interest to see whether the zeros are more consistent with no darkening or with $\frac{2+3\cos\theta}{5}$, *i.e.* solar darkening nearly.

The ratios of the zero points should be

For no darkening : 1 : 1·83 : 2·66.

For $\frac{2+3\cos\theta}{5}$: 1 : 1·76 : 2·52.

The observed values are

He 4387 : 1 : 1·65 : 2·40.

Mg^+ 4481 : 1 : 1·71 : 2·24.

The observed values are lower than either no darkening or $\frac{2+3\cos\theta}{5}$ warrant, and possibly indicate greater darkening than that assumed, but one cannot press them far since there is a probable error of something like 10 per cent., but one can say that they are in favour of the law of darkening assumed.

This is the only star so far examined where this test can be attempted. It is idle to try it with only two zeros for a given line available, and these are the first lines examined to show three well-determined consistent zeros.

Algol.—This star affords a critical test of these methods. There exists a good independent determination of the rotational speed of the brighter component by direct observation of the Doppler displacement at the limb at times near total eclipse * and also an exceptionally well-determined profile, for the lines Mg^+ 4481, has been published by Struve and Elvey in *M.N.*, **91**, 663, 1931. It is there given graphically on p. 666, and is reproduced here in fig. 6 (see p. 517).

Applying the analysis of line shape to this profile we find the following set of zeros, and corresponding velocities :—

		$10^4\beta$		
u	z_1	z_2	z_3	z
0·347	**1·67**			
0·518	1·12	**1·96**		
0·793	0·73	1·28	**1·83**	
0·858	...	...	1·70	(2·19)
0·976	...	...	1·49	(1·94)

* Rossiter and McLaughlin, *Ap. J.*, 60, 22, 1924.

The velocities in any one column of this table are these obtained by supposing the zeros concerned are the first (column z_1), second (column z_2), etc., rotational zeros.

It is clear that the three zeros in clarendon type are possible rotational zeros, within the permissible error of about 10 per cent. Either of the two in brackets might be the fourth. The possible alternative of the first zero being a natural one is rendered unlikely by the fact that the line shows rather too much distortion for this: it could scarcely, however, be ruled out on the strength of this alone, but since we are here only concerned with this line as a test object we know *a priori* this alternative need not be considered;

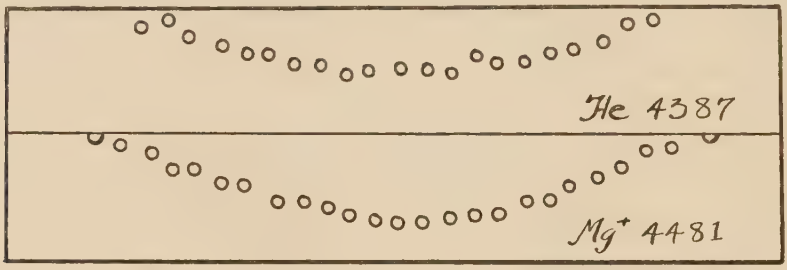

FIG. 5.—39 λ *Orionis.*

The scale of absorption is such that the distance between the horizontal lines represents 20 per cent. of the continuous background.

presumably tests on another line would eliminate it. The mean rotational speed then works out at $10^4\beta = 1{\cdot}84$, or $V = 55$ km./sec. $\pm$ 3, in excellent agreement with the direct determination of 55 km./sec.

Struve and Elvey find a value of 60 km./sec., and allow a margin of error of about $\pm$ 15 km./sec. This, however, is obtained on the assumption of a line unchanged in form over the disc with no darkening, and should be compared with a value of 50 km./sec. obtained by this method on that assumption, or augmented to about 66 km./sec. for comparison with the value of 55 km./sec. obtained here; thus their assumed original line (given on p. 666 of the paper cited above) cannot be very seriously in error.

We have attempted to compute $I(\zeta)$ from the observations and our value of β by Method V of Paper I. As there anticipated, there is some difficulty in applying the method when the zeros are not accurately known. We have attempted to overcome this trouble (arising from the singularities in $\dfrac{\int_{-\infty}^{\infty} O(t)\cos utdt}{G(u\beta)}$ due to "misplacing" of the zeros) by computing the error approximately and applying a correction by shifting the u axis slightly and adjusting the zeros of $\int_{-\infty}^{\infty} O(t)\cos utdt$ to agree with those of $G(u\beta)$, as indicated by a development of the theory that will be published very shortly.

Fig. 6 shows the form of $I(\zeta)$ computed in this manner (assuming no darkening and the appropriate value of β corresponding to 51 km./sec.), and the original assumed by Struve and Elvey. It is seen that the assumed

original is not far out, it is rather too high and too narrow. To offset this a higher value of the velocity is assumed by Struve and Elvey, namely, 60 km./sec. In the same diagram is shown the distorted form corresponding to both original profiles with their corresponding velocities. There is little difference in the distorted forms, while ours mimics the observed shape rather better ; the other is a better fit near the centre of the line. The point is that by inspection of the zeros of $\int_{-\infty}^{\infty} O(t) \cos utdt$ we are able to say at once that *no* original profile will yield the observed form under the hypothesis of no darkening. If we assume, however, that there is no darkening (or any

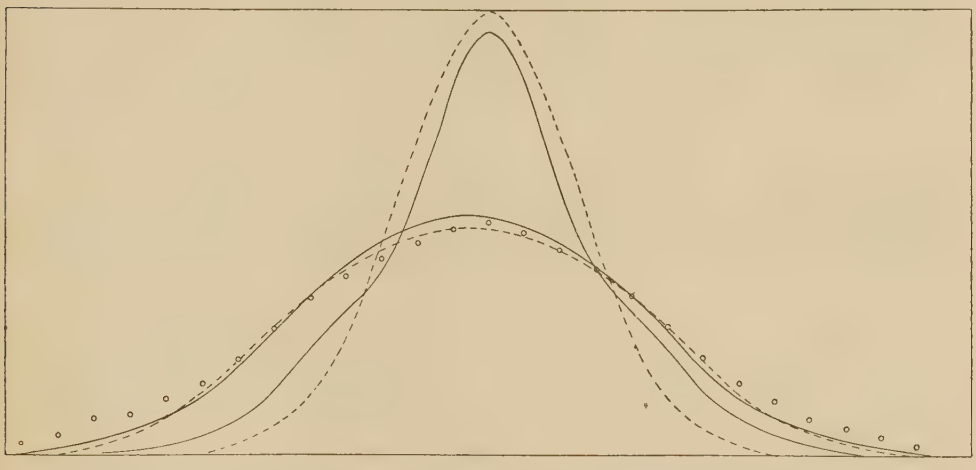

FIG. 6.—*Algol* (Mg^{+} 4481). *The assumed* (- - -) *and computed* (——) *original profiles, and their respective distorted forms* (- - - V = 60 *km./sec.*) *and* (—— V = 51 *km./sec.*) *assuming no darkening or change of line form.*

particular given law of darkening) the method of shape analysis yields a form of $I(\zeta)$ that best accords with this assumption, and the form of the original is then determined. If the zeros are inconsistent with the assumed law of darkening, the "averaging" or smoothing process used to adjust them to fit the assumed law results in an original which gives the best approximation to the observed form under the assumptions made. Clearly, by varying β, one can improve the fit of the deduced distorted form to the observations, but this is hardly legitimate, the proper procedure is to modify the assumption about darkening and change of form. The difficulty in finding a precise value of β by the method used by Elvey and Struve is that there is a fair margin of choice in the original, and correspondingly in β, for a given degree of approximation to the observations. In the present method this elasticity is avoided, or at any rate very substantially reduced, by the determination of β direct from the observations without assuming an original. It is true that a law of darkening has to be assumed, but so it has in their method, but here it only is assumed, and the variation of β with permissible variations in the law of darkening is comparatively slight.

Looking at the matter from a slightly different point of view, the deter-

mination of β by assuming an original profile and a law of darkening is not very sensitive to the form of the original if the distortion be fairly great, the form of the distorted curve being determined largely by the law of darkening and the area of the original. Thus there exists a fairly wide choice of curves with a given area, the appropriate value of β being the larger the narrower and deeper the assumed original.

The computed original shown in fig. 6 is not of course meant to be the best approximation to the true original that can be obtained; to calculate that would mean using the best law of darkening and change of form, and we think it premature to attempt this. The curve given is merely for comparison with the assumed original, to show (*a*) that the higher speed deduced from the assumed original is because it is deeper and narrower than our curve, (*b*) the excellent judgment shown by Struve and Elvey in assuming a suitable original line, and (*c*) the way in which the new process works and the nature of the approximations involved under various assumptions. Finally, it should be said that the slight asymmetry of the observed line has been ignored by us in computing the original profile on the grounds that it is of little importance for the present purpose. Further, no attempt has been made to get the *best* fit of $O(\zeta)$ with the observations. This could have been achieved by attaching greater weight to the first zero of $\int_{-\infty}^{\infty} O(t) \cos ut$, but we are not specially concerned to do this. Actually the position of the rotational zeros for Algol indicates rather greater darkening than has been assumed. This agrees with the suggestion offered by the examination of 39 λ Orionis.

Discussion of the Results.—The shapes of the observed line profiles show when analysed by the method used the elements of shape to be required if they are due to rotational distortion. The internal concordance of the values of β for any one line is striking, and the values from different lines in a given star agree within 10 per cent. or so of the mean value.

Moreover, the values obtained are consistent with a likely law of darkening similar to that known to be true for the Sun.

In view of these results it is hard to avoid the conclusion that these stars are rotating, and that we are able to determine their rotational speeds with remarkable precision, and, further, that their surfaces differ little in their properties from the solar surface. There seems, now that a method exists for finding rotational speeds from the observed line profiles only, without *a priori* knowledge of the undistorted line, a very reasonable prospect of the opening up of a new field of astronomy, of precision at least as good as that of the comparable field of radial velocity determination.

Already observation is able to yield surprisingly good results, and an improvement which would reduce the errors of observation from a few per cent. to under 1 per cent. is by no means improbable (the improvements introduced by Shane should alone lead to a considerable reduction of the errors) with correspondingly great benefit in this work, both in the extension of the field and improvement of accuracy.

One speculation may perhaps be permitted. Why are these broad

shallow lines indicating great rotational speeds found mostly in early-type stars? It seems at least possible that the reason may lie partly in the effect itself. If a spectrum have narrow lines in it when the star is not rotating, the effect of rotation can easily be to blur them into invisibility, so that only the lines which originally were strong (wide and deep) remain. This would presumably lead to the classification of the spectrum as of a type earlier than it really is. An examination of the colour temperature of the star concerned should settle the matter, but for the particular stars described here that is unknown to us. In any case the possibility exists, and it will be of interest to compute the appearance of a section of say the solar spectrum for high rotational speeds. We hope to do this in the near future. Of course one factor selecting early-type stars is the low apparent photographic magnitude needed to photograph the spectra on the scale of dispersion necessary; this tends to select the brighter early-type stars.

Summary.

The methods of Paper I are applied to line profiles of *He* 4387 and 4471, and *Mg* 4481, observed by Elvey in 44 Ori, 44 Per, and 39 Ori, to test for rotational distortion and to find the rotational speeds.

It is concluded that these stars show rotational distortion, and fairly precise determinations of the rotational speed are able to be made. The line *He* 4471 is probably unsuitable for this purpose in general. Very small lines also are not reliable.

The methods of Paper I work well with actual stars, and a new field of astronomical measurement appears to be opened up—one of considerable precision with but a slight improvement in the accuracy of the observations.

The star Algol used as a critical test case shows the methods developed to be practicable and accurate.

The theoretically possible test for the law of darkening is found to be hardly practicable for most stars with the present accuracy of observation, but the observations of 39λ Ori afford a test and are consistent with the law $\frac{2+3\cos\theta}{5}$, rather than with no darkening.

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ON MILNE'S THEORY OF WORLD STRUCTURE.

W. O. Kermack and W. H. McCrea.

1. *Introduction.*—A new approach to the problem of the expansion of the universe has lately been given by E. A. Milne.* He finds a simple explanation of the recession of the extragalactic nebulæ in the fact that an isolated cloud of moving particles in ordinary space will disperse in such a

* E. A. Milne, *Zs. für Astrophysik*, **6**, 1–95, 1933. Referred to as M.

manner that ultimately those with the highest velocities will be furthest from the centre and moving outwards. The circumstance that gives interest and weight to the theory is that he can show that a space-velocity distribution exists such that, consistently with the principle of special relativity, it appears the same as viewed by all observers, or by all of a particular set of observers. In the following communication we examine the relation of this theory to the general relativity theory of non-static universes. We also attempt to answer some of Milne's criticisms of that theory.

Since this paper was written a simple and fair-minded summary of Milne's theory has been given by McVittie.* In raising the question of the distinction between the world-views of the type of observer which he classes as A, and other observers such as he classes as B, he broaches a subject which is vital to our discussion. His class A are our fundamental or privileged observers.

2. *Discussion of Milne's View-point.*—We can readily admit Milne's view in regarding space time merely as a convenient map for the description of phenomena. This is the best way of regarding the space time of general relativity. But from this standpoint it is just a matter of mathematical convenience as to whether we choose to map events in the space time of general relativity, or in the space time of special relativity as is done by Milne. Both methods are always possible, but of course the statements of the natural laws governing phenomena in the two representations will be different.

There is one circumstance, however, which makes the correspondence between the two representations more close in the special problems with which we have to deal than will be the case in general. For Milne uses the space time of special relativity described by the metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \tag{1}$$

where we set the velocity of light equal to unity, while the space time for a homogeneous isotropic universe in general relativity is †

$$ds^2 = dt^2 - R(t)^2 d\sigma^2, \tag{2}$$

where

$$d\sigma^2 = \gamma_{\alpha\beta}(x^1, x^2, x^3) dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3) \tag{3}$$

represents a space of constant curvature k. We can further take, without loss of generality, $k = 1$, 0, or -1 for the cases of positive, zero, or negative curvature. The spatial section of (1) has a euclidean metric,

$$dx^2 + dy^2 + dz^2, \tag{4}$$

while that of (2) has a metric proportional to $d\sigma^2$. Now by a known theorem ‡ the manifold of constant curvature (3) can always be represented *conformally*

* G. C. McVittie, *Nature*, p. 533, 1933 April 15.

† H. P. Robertson, *Reviews of Modern Physics*, 5, 62–90, 1932. This review of "Relativistic Cosmology" supplies a convenient summary of results on the general relativity treatment of the expanding universe.

‡ T. Levi-Civita, *Absolute Differential Calculus* (English translation, 1927), p. 246.

on the euclidean manifold (4), *i.e.* there exists a function $u(x, y, z)$ such that

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{u^2}. \tag{5}$$

Hence in any small neighbourhood the two maps correspond without distortion.

3. The difficulties raised by Milne (M., § 24, p. 17) concerning expanding space and the number of spatial dimensions are hard to understand, but seem to arise from his separate consideration of the spatial part of the continuum. For he says "the objections . . . against *expanding* spherical space have no force in connection with Einstein's static spherical space" (M., p. 22, footnote). But the latter form of space time is best mapped as the four-dimensional surface of a fixed cylinder in five euclidean dimensions, while the expanding universes can be mapped as the surfaces of other fixed figures of revolution in the same euclidean space. The axis of revolution is in all cases the time direction. Looked at in this way there is no difference in kind between static and non-static universes to give one an *a priori* recommendation above the others. We defer to § 9 some remarks on absolute velocity and on cosmic time.

4. *Milne's Two Postulates.*—The two postulates which form the basis of Milne's work are (*a*) that the velocity of light is the same constant for all observers, (*b*) that the history of the universe must appear the same to all observers.

Milne's theory is more easily understood if it is realised that in it there are virtually two types of particles or observers. The first type consists of a group of privileged or fundamental observers in reference to whom Milne's second postulate is accurately fulfilled. In other words, each of these fundamental observers sees exactly the same picture of world events as every other fundamental observer. These observers start from the same point at the same time with uniform velocities relative to each other. If we take a Minkowski representation of the universe the world-lines of these fundamental observers are represented by a pencil of straight lines the vertex of which is arbitrary, and provides Milne's space time origin.

In Milne's general statistical solution there is introduced a second type of particles or observers which are such that they allow the fundamental observers still to have pictures of the universe fulfilling the second postulate, although they do not themselves have the same series of views as the fundamental observers would have.

It is clear that the world consisting of the fundamental observers alone is one which satisfies rigorously both of Milne's postulates. It also seems clear that this is not the only solution conforming to the two postulates, because, for example, if we take the Minkowski picture and choose in it an arbitrary point and an arbitrary curve, then the aggregate of lines obtained by rotating the curve symmetrically in all possible directions about the point would satisfy the two postulates.

Milne's statistical solution yields a space-velocity distribution of particles which appears the same to all the privileged observers. It does not, however, belong to the type of solution we have just described, nor does it appear

strictly to conform to the second postulate as Milne has formulated it. For in applying it to the actual universe we should have to suppose an observer to accompany each particle, in which case it will clearly be no longer satisfied by *these* observers. It is not even satisfied in general by an observer having the mean velocity of particles in his own neighbourhood.

Since the publication of his paper Milne * has pointed out an error which invalidates his original conclusion that all particles move in straight lines in his statistical solution. Even in this latter case, however, it is clear that his postulate is not satisfied by all observers. For, in his discussion of this point (M., pp. 24, 25), his observers A, B have velocities prescribed in advance, and cannot "adopt" different velocities such that each *sees* himself as the centre of the universe for all time. This adoption of a velocity would mean the transformation of the observer's actual world-line into one passing through Milne's origin of space time, *i.e.* the transformation of the observer into one of his fundamental set of observers starting from the same point with uniform relative velocities. Milne's argument serves to show that in the case discussed each observer can at any instant choose a system of space time co-ordinates, with himself as origin, to which he can refer his observations, and such that the aggregate of mathematical descriptions of the universe so obtained by him will be the same as the aggregate for any other observer. But each observer will by no means *see* the same series of pictures of the universe.

In order that Milne's postulate should be strictly fulfilled the problem to be solved is: To find a set of world-lines such that the series of pictures of the universe seen by each of the *corresponding* observers is the same. In Milne's investigation this is satisfied only by his fundamental set of observers amongst themselves, the corresponding set of world-lines being then a family of straight lines passing through the origin.

Milne's second solution is his *hydrodynamic* one, in which he finds a *systematic* motion consistent with his two postulates. This motion is again just that of his fundamental observers. Since this is the only one given by Milne strictly satisfying his postulate, we shall confine attention to it in the following section.

5. *Comparison with the "Expanding Universe" Solution.*—We shall show that Milne's hydrodynamic solution is equivalent to a special case of the general relativity solutions classed as expanding universes.

We write the metric (1) used by him as

$$ds^2 = dT^2 - dx^2 - dy^2 - dz^2. \tag{6}$$

In polar co-ordinates with

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta,$$

we have

$$ds^2 = dT^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{7}$$

If we apply the further transformation

$$T = t \cosh\chi, \qquad r = t \sinh\chi, \tag{8}$$

* E. A. Milne, *Zs. für Astrophysik*, **6**, 244, 1933.

(7) becomes

$$ds^2 = dt^2 - t^2(d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2). \tag{9}$$

We now see that (9) is a special case of the metric (2) with

$$R(t) = t, \qquad k = -1.$$

In the general theory of the expanding universe we consider particles fixed in the spatial co-ordinate system. If now we have particles fixed in the (χ, θ, ϕ) co-ordinates we have just exactly Milne's fundamental system. For, from (8), if χ = constant, then r/T = constant, or

$$\frac{dr}{dT} = \frac{r}{T}, \tag{10}$$

which is just Milne's velocity law. Also the assumption of isotropy and homogeneity ensure that his fundamental postulates are satisfied.

6. *Density.*—The density ρ and pressure p, associated with the metric (2) are *

$$\kappa\rho = -\lambda + 3\frac{(k + R'^2)}{R^2}, \tag{11}$$

$$\kappa p = \lambda - 2\frac{R''}{R} - \frac{(k + R'^2)}{R^2}, \tag{12}$$

where κ is the constant of gravitation and λ is the cosmical constant. For the moment we assume $\lambda = 0$.

If we put $k = -1$, $R = t$, we obtain $\rho = 0$, $p = 0$, as we must, of course, for flat space time. So the metric (9) from the standpoint of general relativity cannot correspond *exactly* to Milne's universe with a non-vanishing density of matter. But we can get a correspondence in the following way.

The equation $p = 0$, which must hold in the universe under consideration, being without random motion, gives

$$2RR'' + k + R'^2 = 0$$

or

$$RR'^2 + kR = A, \tag{13}$$

where A is a constant.

Taking $k = -1$, and assuming $A << R$, we have approximately

$$\frac{dt}{dR} = 1 - \tfrac{1}{2}\frac{A}{R},$$

giving

$$t - t_0 = R - R_0 - \tfrac{1}{2}A \log\frac{R}{R_0}, \tag{14}$$

where $R = R_0$ when $t = t_0$, and there is no loss in generality if we take $R_0 = t_0$. Then from (14), to the first order in A,

$$R - R_0 = t - t_0 + \tfrac{1}{2}A \log\frac{t}{t_0}. \tag{15}$$

* Robertson, *loc. cit.*, equation (3.2).

This gives, from (11),

$$\kappa\rho = 3A/t^3. \tag{16}$$

We can therefore regard the metric (9) as a zero approximation to that for a universe in which the density is given by (16). But *this is exactly the density law in Milne's universe.* For his density law (M., equation (139))

$$\rho_M = \frac{BT}{(T^2 - r^2)^2} \tag{17}$$

reduces, with $r = 0$, to

$$\rho_M = B/T^3, \tag{18}$$

and with $r = 0$ we have $T = t$.

Therefore in our case and in Milne's case an observer at a given point measures the same density, when we adjust the numerical constants. This is sufficient to show that the density laws are the same. We can, however, recover Milne's form (17) by writing (16) as

$$\kappa\rho = \frac{3A}{(T^2 - r^2)^{3/2}}, \tag{19}$$

which is the density in our co-ordinate system in which the matter is at rest. In Milne's system it has the velocity $V = dr/dT$ given by (10), and so by a Lorentz transformation on the volume element

$$\rho_M = \frac{\rho}{\sqrt{1 - V^2}} = \rho\frac{T}{\sqrt{T^2 - r^2}} = \frac{3A}{\kappa}\frac{T}{(T^2 - r^2)^2}. \tag{20}$$

It should be noticed that we do not take account of change of mass with velocity, since Milne is concerned only with numbers of particles, and not *mass* densities.

The conclusion of this section is then: *Milne's hydrodynamic solution*, which, as we have seen, is the only solution of his which strictly obeys his postulates, *is included as a limiting case, when the density tends to zero, and* $\lambda = 0$, *amongst the expanding universes of general relativity.*

This special case of the general relativity solution has, of course, all the essential features of Milne's model. In particular it has no equilibrium configuration, and a finite length of time (see § 12 below) after any given instant it will be expanding, and not contracting. This latter characteristic was one of the attractive features of Milne's theory.

7. *Velocity of Recession.*—The usual method for calculating the Doppler effect gives for a light signal emitted at time t_0 and observed at time t_1

$$\frac{\text{Frequency observed}}{\text{Frequency emitted}} = \frac{t_0}{t_1}. \tag{21}$$

For, along a radial light track we have

$$0 = dt^2 - t^2 d\chi^2,$$

giving

$$\chi_1 - \chi_0 = \int_{t_0}^{t_1} \frac{dt}{t},$$

where χ_0, χ_1 give the fixed positions of emission and absorption. Varying this we have

$$0 = \frac{\delta t_0}{t_0} - \frac{\delta t_1}{t_1},$$

which leads to (21).

8. *The Cosmical Constant.*—It is interesting to inquire if Milne's solution can be the limiting form of one with $\lambda \neq 0$. Putting $\rho = 0$ in (11) we have

$$R'^2 + k - \frac{\lambda}{3} R^2 = 0,$$

from which

$$R'' - \frac{\lambda}{3} R = 0. \tag{22}$$

Substituting in (12) this gives $p = 0$ identically. Thus the limiting case of zero pressure and density can be got with any value of λ. But from (22) we see that the case of unaccelerated relative motion $R'' = 0$ requires $\lambda = 0$.

The limiting case of zero density with $k = 0$, $\lambda \neq 0$ is de Sitter's universe. This can therefore from one point of view be regarded as belonging to the general type of solution considered by Milne in which expansion is a purely kinematic phenomenon which "has nothing to do with gravitation in the first instance" (M., p. 83).

9. *Cosmic Time.*—Milne criticises the concept of "cosmic time" as used in general relativity (M., § 23, p. 16), and asks which observers actually keep this time. The relation of cosmic time to the time used by him is clearly shown in the foregoing discussion. The answer to his question is that the cosmic time of an event is the proper time of that one of his fundamental observers who experiences the event. Cosmic time is thus implicit in his own description of phenomena, in that it thus singles out one particular time reckoning as specially suited to describe a given event. Otherwise stated, in Milne's model the cosmic time of the event is its interval distance from the origin of space time.

Milne, however, regards the fact that all observers assign the same cosmic time to an event as contrary to the spirit of relativity. But cosmic time is largely a mathematical convenience. No co-ordinate has the physical properties of time applied universally. The above treatment has shown that the *same* universe can be described in Milne's way or alternatively by use of a cosmic time, but the *experience* of any observer in it is, of course, quite independent of the system of co-ordinates one may choose to employ.

Neither does the idea of cosmic time go against the principle of special relativity, as Milne appears to suggest. For, at any given point of space time only one observer will have cosmic time as his proper time. All other observers whose world-lines pass through that point will choose their proper times in agreement with the principle of special relativity applied locally. It should be emphasised that in a gravitational field this principle has a meaning only when applied in the neighbourhood of a point. General relativity provides a means of dealing with larger regions, and cosmic time

emerges only when we deal with the smoothed-out model of the whole universe.

It should also be mentioned that special relativity denies a meaning to absolute velocity only in the space to which it applies. This is the *flat* space in the neighbourhood of any point tangential to the actual space. In relation to the entire curved universe, particular velocities may be peculiarly associated with particular points.

10. *Relaxation of Milne's Postulate.*—Instead of taking the strict form of Milne's second postulate as holding for every observer, in applying it to the actual universe we should require it to hold for *an observer on any particle having the mean velocity of all particles in its neighbourhood.* In this form, however, it will be satisfied in all "expanding" universes of the form given by (2) by observers fixed in the (x^1, x^2, x^3) co-ordinates. For the derivation of (2) ensures that there is no resultant flow relative to this co-ordinate system. Therefore, if the pressure does not vanish, the mean of the random motions responsible for it is zero relative to this system. Further, as before, the conditions of homogeneity and isotropy ensure the satisfaction of the postulate for the "fixed" observers.

11. *Other Solutions.*—What has just been said leads us now to consider the possibility of solutions of Milne's whole problem, other than the particular one discussed above.

We have to recall that he is careful to distinguish between his postulate of the invariance of the velocity of light with the *deduction* of the invariance of $dt^2 - dx^2 - dy^2 - dz^2$, and the general relativity postulate of the invariance of a Riemannian metric. But the latter is exactly equivalent to the postulate of the constancy of the velocity of light. Further, we have just seen that Milne's second postulate is satisfied by all observers with fixed spatial co-ordinates in any of the metrics (2). Hence every non-static universe of this form provides a rigorous solution of Milne's problem as formulated in § 4 above, and modified in § 10. The truth of the converse remains a matter for further investigation.

12. *Time Scale.*—Milne obtains as the time since the universe started to expand a period of the order of 2×10^9 years (M., p. 9). This is with neglect of gravitation. We can, however, allow properly for the effect of gravitation from the general relativity standpoint by taking the exact solution of (13) in place of the approximation (14). It is for all $R > 0$, $A > 0$

$$t = R\sqrt{1 + \frac{A}{R}} + A \log\left(\sqrt{A + R} - \sqrt{R}\right) - \tfrac{1}{2} A \log A, \qquad (23)$$

where $t \to 0$ when $R \to 0$, giving the time since the expansion started at $R = 0$. Milne's case corresponds to the limit when $A \to 0$, namely, $t = R$. We can show in general that when $A \neq 0$, the right-hand side of (23) is less than R, i.e. *the effect of gravitation is to decrease the time of expansion.* This can also be seen directly from (23); for, if $R << A$, we have approximately

$$t = \tfrac{2}{3} \frac{R^{3/2}}{A^{1/2}} << R,$$

showing how the effect of gravitation is to speed up the expansion when R is small.

Physically this may be interpreted as follows : If we imagine the present velocities reversed the particles will move closer and closer together and the velocities will gradually accelerate and they will finally fall rapidly together in a shorter time than if there were no gravitation. If the particles start at a point and separate from each other, then in their own gravitational field this requires higher velocities to begin with in order that, after they have scattered, the observed velocities and positions should be attained, than the initial velocities required in the absence of gravitation.

Milne's time of 2×10^9 years would generally be considered too short for the "age of the universe," being actually of the order generally assigned to the "age of the earth." We now see that the allowance for gravitation makes the case worse for this particular model. But if we regard it as a special case of the general relativity solutions we may, by passing to another one, get a longer time scale. For instance, Eddington * considers the 10^{10} years calculated for the Lemaître universe may be adequate.

There is, however, the further circumstance to be noted that Lemaître's universe is the only one with a definite initial state. In the one we have been considering there is no special reason for dating the age from $R = 0$.

While (23) provides the strict solution for our model for all $R > 0$, it should be pointed out that the physical assumption $p = 0$ would not hold down to $R = 0$. But we shall not go further into this.

13. *Observational Test.*—There is one possible observational test of Milne's model to which attention has not yet been drawn. For (17) gives a definite variation of apparent density with apparent distance (though r is not necessarily the co-ordinate in which "distance" will be measured). This it should be feasible to test by sufficient observations on the extra galactic nebulæ.

14. *Gravity in Milne's Treatment.*—Gravitation appears to feature in Milne's paper in two different guises. In its effect in removing singularities in density, and in the concept of velocities of escape, it is thought of apparently as something extraneous, in the nature of Newtonian gravitation, to be added to the simple theory. From our point of view these effects are allowed for by use of the methods of general relativity, Milne's flat space being just the limiting case of zero gravitation.

Milne, however, expresses the hope that it will be possible to *derive* the phenomena of gravitation from his treatment; in other words, that the motions allowed by his two postulates will be just those consistent with the "law" of gravitation.

We can show that the latter possibility does not arise at any rate in Milne's statistical solutions. For, from the correction to his original paper already referred to, it follows that there are different possible solutions of his statistical problem. The superposition of any two of these associated with a common set of privileged observers yields a further solution. Hence the motion of any particle belonging to the one system is unaffected by all the

* A. S. Eddington, *The Expanding Universe*, p. 82, 1933.

particles of the second system. That is, the gravitational effect of these particles upon it is evidently neglected in obtaining the solution, and so the phenomena of gravitation cannot be predicted by these means.*

We cannot apply the same argument to rigorous solutions of the problem as stated in § 4, since two solutions of this form of the problem cannot in general be superposed to give a further solution. For example, one solution is provided by an infinite family of equally spaced parallel straight lines, since in solving the purely kinematical problem we are not concerned with difficulties of infinite gravitational potentials. If this is superposed on the family of straight lines passing through the origin, it is not true that an observer having a member of one family as his world-line will get the same series of world-views as one having a member of the other family as his world-line. Thus the two solutions taken together do not give a solution, and so we cannot argue as before that the theory will not yield the phenomena of gravitation. This problem requires further study.

There is one further point which may be mentioned in this connection. Milne gets as the only solution of his "hydrodynamical" problem the one with unaccelerated motion. That this is the case has nothing to do with gravitation, as suggested by Milne (M., p. 77), but arises from the fact that his observers are supposed *a priori*, in applying the Lorentz transformation, to have uniform velocities relative to each other. If they were allowed to move with his fluid with relative acceleration then, presumably, other solutions would make their appearance, corresponding to other types of expanding universe. The argument that each observer sees himself as the centre of the universe, and is therefore unaccelerated, is not valid. It does follow that he is not accelerated in the frame of reference which he himself naturally chooses, but it does not follow that he is unaccelerated in the frame of reference which any other observer naturally chooses in which he will now appear to be eccentrically placed. In fact, every expanding universe of general relativity provides an example of a case where each observer sees himself as centrally placed, but in which all other observers in general appear to him to possess radial accelerations.

Summary

We examine the relation of Milne's theory to the general relativity theory of the expanding universe. While admitting his general philosophical standpoint with regard to space time, we venture to think that his criticism of the concept of expanding space is illusory. We then point out that his postulate concerning the equivalence of all observers is not strictly obeyed by his *statistical* solution, but only, in his treatment, by his *hydrodynamic* solution. In the second section, §§ 5–13, we show that the latter solution can

* Mathematically, this means that different values of ψ in M., equation (50), can be superposed. This does not remain true when Milne imposes the condition of his § 9 (M., p. 51). Here, however, he seems to have sacrificed the purely *statistical* nature of the solution, in assuming that a particle with given velocity and position, at a given instant, possesses a *uniquely* determined, and not merely statistically determined, acceleration.

be regarded as the limit, when the effect of gravitation tends to zero, of one of the non-static universes of general relativity. *An observer in this universe*, in the limiting state, *will have precisely the same set of experiences as an observer in Milne's model.* Milne is therefore studying exactly the same natural phenomena as those studied by general relativity. In addition, by thus showing how his own model could be described by the use of a cosmic time, this answers his objection to the introduction of cosmic time as being a retrogression from the principle of special relativity, which principle he has employed to derive the time of his observers. We discuss the problem of the time scale in Milne's work, and conclude the section by indicating that the observed spatial distribution of matter should provide a test of the applicability of his model.

The concluding section examines the relation of gravitation to Milne's treatment. We show how his statistical study cannot reproduce the observational effects of gravity. If, however, the result that Milne's work provides the limiting case where gravitation tends to zero be accepted, then by passing to the general relativity treatment of the problem we can in fact make proper allowance for the effects of gravitation. In this view every solution of the general relativity problem satisfying the required conditions of homogeneity and isotropy does actually provide also a rigorous solution of Milne's problem.

Milne starts from the hypothesis that the dispersion of the material universe is essentially a kinematical phenomenon. In so far as this is the case it should be automatically taken account of by the methods of general relativity properly applied. But this does not seem to have been fully realised previously ; and the great value of Milne's work seems to lie in the fact of its showing that this simple effect must be taken into consideration in the *interpretation* of the predictions of general relativity. Though it is contrary to the strict spirit of this theory to distinguish between gravitational and kinematical effects, it may be useful for gaining physical insight into phenomena roughly to make such a distinction. Then Milne's work renders evident which features are to be ascribed to the kinematical effects.

Another important deduction to be drawn from Milne's theory is that great caution is necessary in making any final conclusions, in our present state of knowledge, about the structure of the cosmos. He has at least indicated the possibility of approaching the problem from a distinct and novel point of view. The need for caution is further emphasised by the consideration that the general gravitational equations of the accepted type have been verified at most only in respect of their local applicability. It now seems clear that the red shift in the spectra of the distant nebulæ is capable of explanation independently of these equations. If these equations are only approximations, then conclusions drawn from them about the cosmological problem may be entirely misleading. From this point of view great interest attaches to the recent theory proposed by Lanczos * in which the gravitational equations are fundamentally modified.

* C. Lanczos, *Zs. für Physik*, **73**, 147–168, 1931.

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GREENWICH PERSONALITY AND THE EQUINOX CORRECTION.

J. K. Fotheringham, D.Litt.

In 1925 Mr. Cullen * suggested that an apparent discontinuity of $0^{s}.04$ in the Greenwich observations of the Sun's right ascension was due to the introduction of the moving wire. In the following year Dr. Jones † accepted this correction. In the same year ‡ I suggested that the change in the observed right ascension was largely due to a change in observers. I pointed out that "Mr. Witchell's Right Ascensions were in 1914 markedly less than those of any other observer, and he made about a third of the whole number of observations of the Sun in that year. In 1915, using the new method, he was in line with his colleagues. 1915 was, however, characterised by the appearance of a new observer, Dr. Jackson, who had a marked personality in the opposite direction, and in 1915 and 1916 he took the largest share in the observations of the Sun—rather less than a third in each year." I inferred that there was no reason for regarding the change as more important than the changes in *personnel* made on other occasions.

In 1927 Dr. Jackson and Mr. Cullen § replied and gave a table showing the mean corrections to Newcomb's tabular right ascension of the Sun as found by each Greenwich observer for each year from 1901 to 1925. They also exhibited the mean correction from all observations, irrespective of the distribution among different observers, the correction to true longitude for each year as obtained from the declinations, and the deduced correction to Newcomb's equinox. They inferred that a change of personality amounting to $0^{s}.031$ occurred on the introduction of the "impersonal" micrometer, and that there had been little or no change in the correction to Newcomb's equinox. This conclusion was not based on the figures for the individual observers, but on the annual means, obtained irrespective of the distribution of the work of observing.

Their conclusion was adopted by Dr. De Sitter || in his discussion of the secular accelerations and the fluctuations of the longitudes of the Moon, the Sun, Mercury and Venus, and in subsequent papers. Like Dr. Jones, he treats it as a permanent difference in Greenwich personality before and after the introduction of the travelling wire micrometer. Dr. Jones now uses this revised correction in the same way.¶

In my opinion this not only vitiates their constants of the Sun's longitude, but also acts as an obstacle to an improved understanding of the relation of the Earth's rotation to the motion of its axis.

The only satisfactory way to determine whether a discontinuity in

* *M.N.*, 85, 1000, 1925.
† *Ibid.*, 87, 9, 1926.
‡ *Ibid.*, 87, 152, 1926.
§ *Ibid.*, 87, 459, 1927.
|| *B.A.N.*, 124, 26, 1927.
¶ *Ann. Cape O.*, 13, 3, 32, 1932.

observed right ascensions is due on the one hand to a change of observers or to a redistribution of work among observers, or on the other hand to a permanent change of personality affecting all observers from the introduction of a change in the method of observing, is to confine our attention to the results derived from those observers who observed continuously both before and after the change. But we cannot judge the effect of the change of method satisfactorily unless we bear in mind other causes producing a change in the observed corrections to the tabular right ascension. Of these, two alone appear to be important—the error in the tabular mean longitude, and the fluctuation in the Sun's longitude produced by the variable rotation of the Earth. Recent studies in the longitude of the Sun show that the mean right ascension was increasing at a rate of about $0^s.003$ per annum at epoch 1915 as compared with Newcomb's tables referred to Newcomb's equinox. The Moon's longitude, when analysed by Cowell periods, attained a maximum, as compared with theory, at period with mean epoch 1916·76, and showed the same value at the following mean epoch 1917·90. It sank till mean epoch 1920·16, then rose to the two former maxima, which it reached at mean epoch 1922·43, and then fell rapidly. Apart, therefore, from the correction to the mean motion, we should expect the observed correction to the Sun's right ascension to increase till 1916 or thereabouts, and after remaining nearly stationary to fall after 1922. As a fact the observations, when corrected for the error in Newcomb's mean motion, show a maximum in 1916. The table given by Dr. Jackson and Mr. Cullen on p. 460 of their paper shows a mean advance of $0^s.005$ per annum from 1901 to 1914, of which we may attribute about $0^s.003$ to mean motion and $0^s.002$ to fluctuation.

Bearing these changes in mind, a glance at Dr. Jackson's and Mr. Cullen's table will show that the introduction of the moving wire increased W.'s and H. A.'s observed right ascensions and diminished R. C.'s and F. J.'s. In fact it affected different observers in different directions, as it did also at the Cape, while at both observatories it diminished the total range of differences of personality. In the accompanying Table I, I confine my attention to the observations made by W., R. C. and H. A., the three observers who observed continuously from 1907 to 1925, except that there are no observations by H. A. in 1911 nor by R. C. in 1918. In these cases I have attributed to the observer the mean between his observed values of the preceding and following year.

$\Delta'\alpha$ is the mean of all observations of right ascension as given by Dr. Jackson and Mr. Cullen, with the addition of the constant $+0^s.005$ from 1915 onwards to reduce all observations to Newcomb's equinox. Mean $\Delta\alpha$ is the mean of the three observers, assigning equal weight to each observer irrespective of the number of observations made by him. Where one of the three meaned values has been obtained by interpolation, I have enclosed the mean in brackets. In the next column I give means of three years of the three observers against the central year of each group. $\Delta\lambda$ is the correction to Newcomb's tabular longitude referred to true equinox as derived from the declinations. It is taken from Dr. Jackson's and Mr.

TABLE I

Year	$\Delta'\alpha$	Mean $\Delta\alpha$	3 Years' Mean $\Delta\alpha$	$\Delta\lambda$	Mean Δ Eq.	3 Years' Mean Δ Eq.
	s	s	s	s	s	s
1907	+·028	+·007	...	−·009	−·016	
08	+·068	+·047	+·037	+·013	−·034	−·035
09	+·046	+·057	+·056	+·003	−·054	−·042
10	+·054	+·063	+·067	+·026	−·037	−·043
11	+·064	[+·080]	+·070	+·043	[−·037]	−·044
12	+·070	+·067	+·073	+·008	−·059	−·054
13	+·060	+·073	+·068	+·007	−·066	−·050
14	+·061	+·063	...	+·038	−·025	
15	+·098	+·078	...	+·043	−·035	
16	+·115	+·108	+·091	+·042	−·066	−·050
17	+·107	+·088	+·101	+·040	−·048	−·067
18	+·112	[+·107]	+·101	+·019	[−·088]	−·078
19	+·117	+·108	+·103	+·010	−·098	−·072
20	+·107	+·095	+·105	+·066	−·029	−·071
21	+·108	+·112	+·105	+·025	−·087	−·064
22	+·122	+·108	+·112	+·032	−·076	−·069
23	+·111	+·115	+·109	+·070	−·045	−·056
24	+·118	+·105	+·114	+·059	−·046	−·051
25	+·119	+·122	+·118	+·060	−·062	−·054
26	+·123	+·126	+·119	+·072	−·054	−·046
27	+·118	+·109	+·115	+·087	−·022	−·032
28	+·119	+·110	+·108	+·090	−·020	−·021
29	+·114	+·106	+·110	+·086	−·020	−·038
30	+·121	+·114	+·111	+·041	−·073	−·036
31	+·117	+·113	+·104	+·097	−·016	−·024
32	+·087	+·086	...	+·103	+·017	

Cullen's paper and represents the mean of all observations. It is of course independent of personality in right ascension observations. "Mean Δ Eq." is $\Delta\lambda$ – Mean $\Delta'\alpha$, and "3 Years' Mean Δ Eq." shows the mean of three successive values of "Mean Δ Eq." against the central year of each group of three years.

The Astronomer Royal, Mr. Greaves and Mr. Cullen have kindly continued the series to 1932, and have corrected an error in $\Delta'\alpha$ for 1922.

It will be seen that the large discontinuity between 1914 and 1915 which appears in the column $\Delta'\alpha$ is not represented in the column "Mean $\Delta\alpha$." In fact there are no fewer than nine larger intervals between successive values as shown in the series. The only figure which lends the least support to Mr. Cullen's and Dr. Jackson's theory of a permanent discontinuity of $0^s.031$ is the value $+0^s.108$ for 1916. The run of the series suggests that the value for 1914 is accidentally low and that for 1916 accidentally high. But, whatever may be the case with 1914, I incline to regard the rapid rise from 1915 to 1916 as real; it was found by all four observers. The only other date in Dr. Jackson's and Mr. Cullen's series in which an advance was found by all observers from one year to the next was from

1903 to 1904. Whichever view we take, it is impossible to found a case for permanent discontinuity on the observed right ascension of one year.

The observed values of the equinox correction ought to provide a better test of discontinuity. The correction to right ascension combines the effect of changes in the Earth's rotation and error in mean motion with observational errors. As the first two of these are common to right ascension and declination results, the observed ΔEq. obtained by deducting $\Delta\alpha$ from $\Delta\lambda$ is free from all complications and is subject to two changes only from year to year, viz. changes in observational errors and changes in the position of the true equinox as referred to Newcomb's equinox. Declination results are generally less consistent than right ascension results, but are almost free from personality. Their accidental errors can be reduced greatly by taking means of three years. When thus meaned, the difference between the right ascension and declination results shows a remarkably continuous series. In fact the mean of the three years preceding the introduction of the moving wire is identical with the mean of the three years following that event. Any continuous effect of personality in right ascension lasting through the three years is of course retained in the three years' mean. The inference therefore is clear that the mean personality of the three observers who observed continuously before and after the introduction of the moving wire suffered no change on the introduction of that wire, and that the apparent change which Dr. Jackson and Mr. Cullen attributed to the introduction of the wire was due solely to a change in *personnel* and a redistribution of work, a change of a kind which has been going on continuously and in no way distinguishes the observations since 1915 from those which preceded them.

At the same time this study of the mean results of the three observers brings out what it is important to prove—that the fluctuation in the Earth's rotation is accompanied by a fluctuation in the motion of the equinox. The equinox correction was clearly changing steadily in a negative direction to 1919 or thereabouts and in a positive direction after that date. As has been seen, the rate of rotation appears to have reached its minimum in 1916. The two motions are found to be concomitant, though there is a real or apparent small difference in their turning-points.

In two papers entitled "Trepidation," published in *M.N.*, **67**, 163, 164, 1926; **67**, 189, 190, 1927, I showed that, after an adjustment of the mean motion of the equinox, the fluctuation in the equinox correction resulting from Greenwich meridian observations of the Sun from about 1850 to 1924 was in the opposite direction to the fluctuation in the right ascension, and maintained a constant ratio of approximately 3 : 5 to that fluctuation. This assumption was found to agree with the results obtained by Newcomb from occultations back to about 1800, but broke down when applied to the eighteenth century. It should be noted that in the second of these papers the computed value of the equinox correction for 1761, printed as $-1''\cdot33$, should be $+1''\cdot33$. The failure of the formula when applied to the eighteenth century is an example of the impossibility of representing a fluctuating quantity satisfactorily by a constant, where we have no knowledge of the

law governing the fluctuation. A formula of this kind may be a good interpolation formula, but it is hazardous to apply it far beyond the range of observations on which it is based.

The close connection between the fluctuation in the rate of the Earth's axial rotation and the fluctuation in the motion of the equinox can be clearly traced in the Greenwich meridian observations of the Sun from 1760 or 1770 onwards. Table II shows the correction to the right ascension of the

Table II

Greenwich Longitudes of the Sun compared with Le Verrier

	Δ R.A.		Δλ		Δ R.A.		Δλ
	″		″		″		″
1750–62	+0·33	1753–57	+0·78	1863–66	+0·31	1889–92	−0·05
65–71	+0·37	58–62	+1·50	67–70	+0·35	93–96	−0·78
72–78	+0·74	65–70	−0·23	71–74	+0·12	97–00	−0·76
79–85	+2·89	71–78	+0·48	75–78	−0·12	1901–05	−0·79
86–92	+1·51	79–85	+1·23	79–82	−0·05	06–14	−0·51
93–97	+1·87	86–91	+0·48	83–88	−0·20		
98–02	+1·02	92–97	+1·12	89–92	−0·44		
1803–06	+0·27	98–03	+0·41	93	−0·57		
07–10	−0·34	1804–10	+0·18	94	−0·93		
11–14	−3·33	12–16	−0·15	95	−0·80		
15–18	−1·99	17–22	−0·41	96	−0·76		
19–22	−0·51	23–28	+0·43	97	−0·86		
23–26	−1·08	29–34	−0·08	98	−0·86		
27–30	−0·42	35–40	−0·12	99	−0·90		
31–34	+0·76	41–46	+0·21	1900	−0·69		
35–38	+1·16	47–52	+0·25	01	−0·42		
39–42	+0·84	53–58	+0·55	02	−0·39		
43–46	+0·15	59–64	+0·03	03	−0·72		
47–50	−0·10	65–70	−0·23	04	−0·17		
51–54	+0·40	71–76	−0·15	05	−0·27		
55–58	+0·36	77–82	−0·90	06	−0·22		
59–62	−0·02	83–88	−0·27				

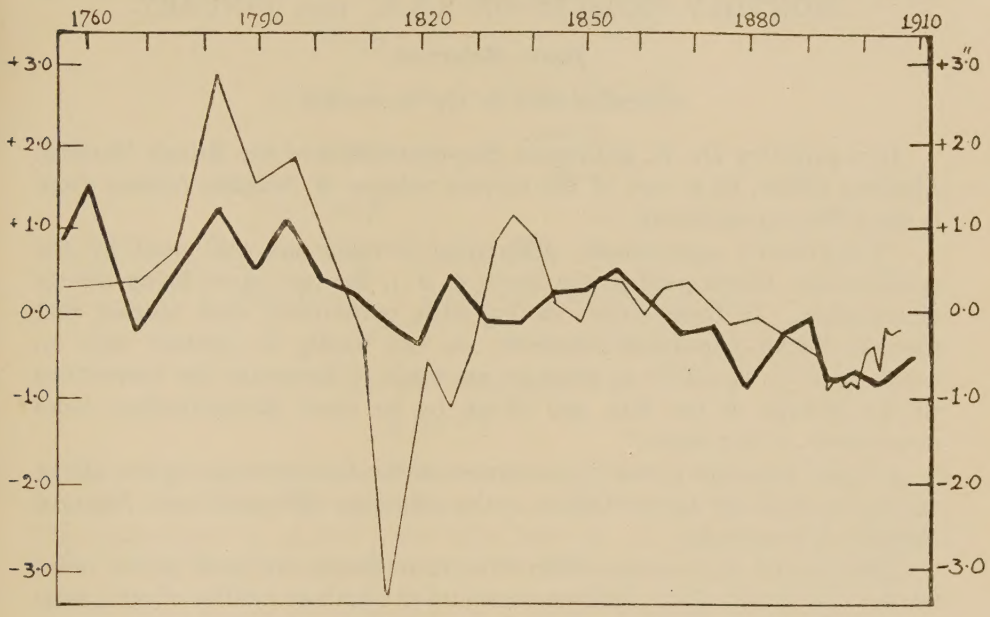
Sun, as given in Le Verrier's tables corrected for certain periodic terms, and referred to Newcomb's equinox, taken from Newcomb's *Astronomical Constants* (1895, p. 22), for groups of years from 1750 to 1892, and the corrections to Le Verrier's uncorrected tables given by Dr. Ross * for individual years from 1893 to 1906.† It shows also the correction to the Sun's longitude referred to true equinox given by the observations in declination, taken from Newcomb's *Astronomical Constants* (p. 30), for groups of years from 1753 to 1892, and from a paper by Sir Frank Dyson and Mr. Cullen ‡ for groups of years from 1893 to 1914. Here again there have been corrections to certain of Le Verrier's periodic terms as far as 1892, but not afterwards. The differences between the corrections introduced into right ascension and declination results respectively are unimportant.

* *A.J.*, 29, 154, 1916.

† For the years 1901–6 the reduction from Newcomb's tables to Le Verrier's is taken from Dr. Ross's tabulation of the Paris results.

‡ *M.N.*, 89, 259, 1929.

This evidence is most easily appreciated with the help of a diagram. The accompanying diagram shows the right ascension and declination results for the central date of each group. It will be observed that throughout the whole range of the comparison the slope of the declination graph is either in the opposite direction to the slope in the right ascension graph



The thin line represents the correction to Le Verrier's longitude of the Sun given by Greenwich R.A.'s.

The heavy line represents corrections to Le Verrier's longitude of the Sun given by Greenwich Declinations.

or is a gentler slope in the same direction. There are only the faintest and briefest traces of the declination graph sloping more steeply in the same direction as the right ascension graph. The inference seems clear that throughout the whole range of observations down to 1906 the fluctuations in the rate of the Earth's rotation, represented by the right ascension results, have been accompanied by fluctuations in the motion of the equinox, yielding an equinox correction which fluctuates in the opposite direction to the right ascension as measured from Newcomb's equinox. That this was so in the Greenwich observations after 1906 has been shown already.

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COMMENTS ON STATEMENTS BY DR. L. J. COMRIE IN MONTHLY NOTICES OF R.A.S., 1933 JANUARY.

James Robertson.

(Communicated by the Secretaries)

In a paper by Dr. L. J. Comrie, Superintendent of the British Nautical Almanac Office, on p. 177 of the current volume of *Monthly Notices* there is the following statement :—

"Chauvenet's approximate differential formulæ are still used by the ephemerides, which publish the limits to 0′·1, the last figure being utterly meaningless. In these more exacting days, particularly with eclipses that traverse thickly-populated countries, we can hardly be content with an accuracy of 3′, especially as attempts are made to determine the corrections to the latitude of the Sun and Moon by an exact determination, from observation, of the limits."

I invite attention to the incorrectness of the first sentence of the above quotation in so far as the Office of the *American Ephemeris* and *Nautical Almanac* is concerned.

Chauvenet's approximate differential formulæ are not used in this office for the calculation of the limiting curves of the path of totality of total solar or annular eclipses, and have not been used for a number of years.

As a result of the recognition of the well-known facts stated in the second part of the above quotation, I introduced formulæ twenty-five years ago by which these curves are calculated to the same degree of accuracy as the central line is calculated.

Accuracy rather than speed has been the chief consideration in the computation of these curves, and no pains have been spared to reach the greatest degree of accuracy attainable consistent with the data on which they are based.

The notation is that used by Chauvenet in his adaptation of Bessel's method in *Practical and Spherical Astronomy*, 1, 436–516.

The formulæ are as follows :—

$$\sin\beta\,\sin\gamma = x - l\sin Q_0$$

$$\sin\beta\,\cos\gamma = y_1 - \frac{l}{\rho_1}\cos Q_0$$

$$\epsilon = \frac{i\cos(Q_0-\gamma)}{\sin 1'}$$

$$\epsilon' = (d_1 - d_2)\cos\gamma$$

$$\rho_2\sin d_2 = \sqrt{1-e^2}\,.\,\sin d$$

$$\rho_2\cos d_2 = \cos d$$

($\frac{i}{\sin 1''}$, $d_1 - d_2$ and ρ_2 may be taken as constant for the entire period of totality)

$$\zeta = \rho_2 \cos(\beta + \epsilon + \epsilon')$$

$$\tan Q_1 = \frac{b' - \zeta f \sin F - \dfrac{a'}{\cos Q_0}}{c' - \zeta f}$$

$$\xi = \sin \beta' \sin \gamma' = x + \sin Q(i\zeta - l)$$

$$\eta_1 = \sin \beta' \cos \gamma' = y_1 + \frac{\cos Q}{\rho_1}(i\zeta - l)$$

$$\zeta_1 = \cos \beta'$$

$$c \sin C = \eta_1 \qquad \cos \phi_1 \sin \theta = \xi$$

$$c \cos C = \zeta_1 \qquad \cos \phi_1 \cos \theta = c \cos(C + d_1)$$

$$\sin \phi_1 = c \sin(C + d_1)$$

$$\tan \phi = [0{\cdot}001465] \tan \phi_1$$

$$\omega = \mu - \theta$$

The angle Q_0 is the complement of the Q used in the computation of duration ; and from the same source are taken the quantities l, b', c' and f. The quantities x, y_1, d_1 and μ are taken from the computation of the central line ; a' has been previously computed and is interpolated to the desired time ; and sin F is practically constant through the eclipse.

After Q_1 has been computed it is substituted for Q_0, and the values of Q and ζ to be used in the final computation are obtained very rapidly as most of the terms are unchanged by the substitution of Q_1 for Q_0. Only near the ends, where great exactness is of no importance, will there be any appreciable change in ζ and Q_1.

With Q exact to hundredths of a minute and ζ to five decimals it is possible to determine the geographical positions theoretically exact to hundredths of a minute of arc, by using x and y_1 to six decimals, and d_1 and μ to tenths of a second of arc.

This method is preferable to many, including Comrie's, that have been submitted to me during the past twenty years, because it is simpler and shorter.

I may add here that the major part of the work of calculating an eclipse is the perfecting of the data on which it is based. The right ascension and declination of the Sun and Moon are checked and carried one place further than their published values, in order to avoid accumulated errors. They are then smoothed so it may be possible to detect errors by differencing, care being taken not to change their values near the centre.

Corrections to the mean longitude of the Moon are obtained from selected groups of the latest occultations obtainable. All observations where the involved angle $(\chi - \rho)$ exceeds 45° are rejected and weights assigned to those retained.

By using these corrections to extend the graphic curves published by Professor E. W. Brown, the path of totality of the eclipse of 1930 April 28 was predicted almost exactly. The southern curve of the eclipse of 1932 August 31, as near as could be determined from photographs taken from an aeroplane by W. M. Browne of the U.S. Naval Observatory, after allowing for the depression on that part of the Moon, was quite close to the predicted position.

Note added by L. J. Comrie

I am glad to be corrected in my inadvertent misstatement. I was misled by the statement in the preface to the *American Ephemeris* "The elements of eclipses of the Sun . . . are given in accordance with Bessel's method, the special forms employed being a modification of those developed in Chauvenet's *Spherical and Practical Astronomy*." It was natural to assume that the modifications referred to were those published by Buchanan (formerly the principal eclipse computer of the A.E. Office) in his "The Theory of Eclipses," which purported to set forth the procedure actually adopted in the A.E. Office. I was not aware of the later improvements introduced by Robertson, more particularly as they have not, to the best of my knowledge, been published anywhere. They are, of course, like my formulæ, based on Chauvenet; it is not necessary to enter into any controversy concerning the relative merits of the alternative methods other than to remark that my formulæ were specifically intended for use with calculating machines.

The success of the recent eclipse predictions made by the A.E. Office is certainly a testimony to the care bestowed on this work.